PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
Volume 27 August 15, 1941 Number 8
Copyright 1941 by the National Academy of Sciences

THE FORM OF GASEOUS CONFIGURATIONS ROTATING WITH NON-UNIFORM ANGULAR VELOCITY

By Zdeněk Kopal

Harvard College Observatory

Communicated June 30, 1941

Equations of the Problem.—Let the free surface of a distorted gaseous configuration be expansible in a series of tesseral harmonics of the form

$$r = a\left\{1 + \sum_{m,n} Y_n^m\right\}, \tag{1}$$

where a denotes the mean radius characterizing the surface in question and the Y_n^m's are tesseral harmonics, with coördinates taken with respect to the center of mass of the distorted body. Let, further, the distortion be so small that quantities of the order of squares and cross-products of the Y's can be neglected. Application of hydrodynamical equations then implies that the Y's must satisfy the equation

$$Y_n^m \int_0^a \rho a^2 da - \frac{1}{(2n+1)a^n}\int_0^a \rho \frac{\partial}{\partial a}(a^{n+3}\, Y_n^m)da - \frac{a^{n+1}}{2n+1}\int_a^{a_1} \rho \frac{\partial}{\partial a}\left(\frac{Y_n^m}{a^{n-2}}\right)da = \frac{c_{m,n}}{12\pi G}\, a^{n+1}\, P_n^m(\phi, \theta), \tag{2}$$

where a_1 specifies the external surface of the distorted configuration; ρ is the density-function; P_n^m's are associated Legendre functions; G denotes the gravitation constant; and the $c_{m,n}$'s are constants which depend upon the nature of distortion.

If the latter is due to the rotation about a fixed axis, the centrifugal force, to the above degree of accuracy, gives rise to one non-zero term $c_{0,2} = \omega^2$, where ω denotes the angular velocity of rotation; thus the only significant harmonic is Y_2^0. In what follows—there should be no danger of confusion—we shall drop the suffices and write simply Y. If so (2) reduces to

$$\frac{5Y}{a^3}\int_0^a \rho a^2 da - \frac{1}{a^5}\int_a^{a_1} \rho \frac{\partial}{\partial a}(a^5 Y)da - \int_0^a \rho \frac{\partial Y}{\partial a} da = \frac{5}{6}\frac{\omega^2}{2\pi G} P_2(\cos\theta) \quad (3)$$

an equation derived first by Clairaut;[1] the more general form (2) is due to Legendre.[2]

Liapounoff[3] has proved that the "well-behaved" particular solution of equation (3) can be expressed as

$$\left\{12\pi G/P_2(\cos\theta)\right\} Y = 3\frac{\omega^2}{\bar{\rho}} + \frac{W}{a^5}\int_0^a \rho \frac{\partial}{\partial a}(\omega^2 a^5 U)da + U\int_a^{a_1} \rho \frac{\partial}{\partial a}(\omega^2 W)da, \quad (4)$$

where $\bar{\rho}$ denotes the mean density interior to a, while U and W/a^5 are the two independent solutions of the equation

$$\frac{a^2}{6}\frac{\partial^2 Y}{\partial a^2} + \frac{\rho}{\bar{\rho}}\left\{a\frac{\partial Y}{\partial a} + Y\right\} = Y, \quad (5)$$

obtained by repeated differentiation of (3) considering its right-hand side as constant. The general solution of (5) then evidently is

$$Y = C_1 U + C_2(W/a^5),$$

where C_1 and C_2 are integration constants. Both U and W are continuous and positive functions of a within the interval $0 \leqslant a \leqslant a_1$, possessing no singularities unless the density-distribution is discontinuous. The values of the integration constants follow from the boundary conditions; from (2) it is easy to see that $Y(0) = 1$; hence we have at once $C_1 = 1$, $C_2 = 0$.

At the surface of the distorted configuration $a = a_1$, and (4) evidently reduces to

$$Y(a_1) = \frac{1}{2}\left\{\omega_1^2 + \frac{3}{h}\int_0^{a_1} \rho \frac{\partial}{\partial a}(\omega^2 a^5 U)da\right\}\frac{P_2(\cos\theta)}{2\pi G\rho_m}, \quad (4.1)$$

where

$$h = a_1^5 U_1\left\{2 + \frac{a}{U}\frac{\partial U}{\partial a}\right\}_{\rho_m},$$

$\rho_m = \bar{\rho}(a_1)$ being the mean density of the configuration as a whole. In this formula, as well as in what follows, the subscript 1 will refer consistently to the surface values of the respective quantities. Remembering

that $P_2 = -(1/2)$ at the equator and $+1$ at the pole, the oblateness $e = (1 - b/a)$ of a slowly rotating configuration becomes

$$e(a_1) = \frac{3}{4}\left\{1 + \frac{3}{\omega_1^2 h}\int_0^{a_1} \rho \frac{\partial}{\partial a}(\omega^2 a^5 U)da\right\}v + \ldots. \qquad (6)$$

where

$$v = \frac{\omega_1^2}{2\pi G\rho_m}.$$

Owing to our neglect of terms of the order of squares and cross-products of the Y's in (2), equation (6) is valid to the first order in v. The integral remaining at the right-hand side of (6) by partial integration becomes

$$\int_0^{a_1} \rho \frac{\partial}{\partial a}(\omega^2 a^5 U)da = (\rho\omega^2 a^5 U)_1 - \int_0^{a_1} \omega^2 a^5 U \frac{d\rho}{da}da. \qquad (7)$$

Uniform Rotation.—If ω = constant, equation (6) can be considerably simplified. For as the reader may easily verify equation (5) can be rewritten as

$$\frac{\partial}{\partial a}\left(a^6 \frac{\partial U}{\partial a}\right) = 6a^4\left\{1 - \frac{\rho}{\bar{\rho}}\right\}\frac{\partial}{\partial a}(aU), \qquad (5.1)$$

the first integration of which can be carried out at once. Integrating subsequently its right-hand side by parts and imposing the boundary conditions we obtain that

$$\frac{1}{a_1^5}\int_0^{a_1} a^5 U \frac{d\rho}{da}da = \frac{a_1}{3}\left(\frac{\partial U}{\partial a}\right)_1 - U_1\left\{1 - \frac{\rho}{\bar{\rho}}\right\}, \qquad (8)$$

which, with the aid of (7), reduces (6) to

$$e(a_1) = \frac{3}{4}\left\{\frac{5}{2 + \frac{a}{U}\frac{\partial U}{\partial a}}\right\}_1 v, \qquad (6.1)$$

which is the form originally derived by Clairaut.[4]

Presuming nothing on the density distribution in the interior of our configuration, we propose to investigate three following models: (*a*) the whole mass of the configuration is condensed at its center (Roche's model); (*b*) the configuration is homogeneous (Maclaurin's model); (*c*) the whole mass of the configuration is confined to an infinitesimally thin surface layer (shell-model). The models (*a*) and (*c*) represent the absolute limiting cases whatever the internal density distribution.

If the whole mass is condensed at the center, then $d\rho/da = 0$ for $a > 0$

and, furthermore, the density at the surface vanishes. Hence (8) yields at once

$$\frac{a_1}{U_1}\left(\frac{\partial U}{\partial a}\right)_1 = 3 \quad \text{and} \quad e = \frac{3}{4}v,$$

in agreement with results of direct integration of equation (5).[5] If the configuration is homogeneous $d\rho/da = 0$ throughout, but the surface density is finite ($\rho_1 = \rho_m$). Hence (8) yields

$$\left(\frac{\partial U}{\partial a}\right)_1 = 0, \quad \text{and} \quad e = \frac{15}{8}v,$$

again in agreement with results of direct integration.[5] If, finally, the whole mass of the configuration is confined to an infinitesimally thin surface shell, $d\rho/da = 0$ for $a < a_1$, but $\rho_1/\rho_m \to \infty$. Equation (8) then breaks down; but (5) reduces evidently to

$$\frac{\partial U}{\partial a} + \frac{U}{a} = 0,$$

which yields at once

$$\frac{a}{U}\frac{\partial U}{\partial a} = -1 \quad \text{and} \quad e = \frac{15}{4}v.$$

Therefore, given v, the superficial oblateness of a slowly rotating configuration whose whole mass is confined to a thin surface shell is twice as large as it would be if the body were homogeneous, and five times as large as that appropriate for a centrally condensed model. Irrespective of the distribution of density in the interior, the values of e/v must fulfill the inequality

$$\frac{15}{4} \geqslant \frac{e}{v} \geqslant \frac{3}{4};$$

while if we restrict the possible density distribution by an assumption that the density is a monotonous function which does not increase outwards, we have

$$\frac{15}{8} \geqslant \frac{e}{v} \geqslant \frac{3}{4}.$$

The latter inequality has first been proved by Clairaut (op. cit.); but the former, more general one, is new.*

Non-Uniform Rotation.—Let us now pass on to configurations for which ω is no longer constant, but depends on a; and consider again the special cases of a homogeneous and centrally condensed model. If the body is homogeneous we obtain from (7) at once that

$$\int_0^{a_1} \rho \frac{\partial}{\partial a} (\omega^2 a^5 U) da = (\rho \omega^2 a^5 U)_1,$$

while for Roche's model we have

$$\int_0^{a_1} \rho \frac{\partial}{\partial a} (\omega^2 a^5 U) da = 0.$$

In either case equation (6) leads to the same oblateness which we found for the respective models set in uniform rotation. Thus irrespective of eventual variation of ω in the interior, the superficial oblateness of such configurations set in slow rotation is completely specified by their angular velocity at the surface and their mean densities—at least as far as squares and higher powers of v are negligible.

Before proceeding further, let us briefly examine whether or not such configurations may rotate with non-uniform angular velocities. Provided that the density depends only on the pressure, the hydrodynamical equations combined with the equation of continuity require[6] that any possible variation of ω should fulfill the following equation:

$$\frac{1}{l} \frac{\partial P}{\partial l} + \rho^2 \frac{\partial \omega^2}{\partial \rho} = 0, \tag{9}$$

where P denotes the pressure and l the distance from the axis of rotation. Now, for any model in hydrostatic equilibrium we have

$$\frac{dP}{da} = -G \frac{M(a)}{a^2} \rho, \tag{10}$$

where $M(a)$ is the mass interior to a. In the envelope of a centrally condensed configuration the density is infinitesimal and, owing to (10), the pressure is constant. Hence from (9) it follows that ω is constant. The envelope of a configuration in hydrostatic equilibrium built up according to Roche's model cannot rotate otherwise than with constant angular velocity.

As to the Maclaurin's model, a fluid mass can be homogeneous only if it is incompressible. If so, the pressure no longer depends on the density and equation (9) breaks down; but the equation of continuity yields at once that

$$\frac{\partial(l_1^2 \omega)}{\partial(x, y)} = 0, \qquad \text{where } x^2 + y^2 = l^2. \tag{11}$$

It follows that ω must be either constant or a function of l only.

If, now, $d\rho/da = 0$ for $a > 0$ and the density vanishes at the surface, we have (cf. equation 7)

$$\int_0^{a_1} \rho \frac{\partial}{\partial a} (\omega^2 a^5 U) da = - \int_0^{a_1} \omega^2 a^5 U \frac{d\rho}{da} da.$$

Unless the functions $\omega(a)$ and $\rho(a)$ are explicitly known the integral on the right-hand side cannot be evaluated directly. If, however, the density is finite in the interior, equations (9) and (10) show that $\partial\omega^2/\partial\rho$ will generally be positive; i.e., ω will increase with the density. If the density increases inwards, we have at any interior point the inequality

$$\omega^2 a^5 U \frac{\partial \rho}{\partial a} > \omega_1^2 a^5 U \frac{\partial \rho}{\partial a},$$

which, with respect to (8), yields

$$- \int_0^{a_1} \omega^2 a^5 U \frac{\partial \rho}{\partial a} da > \frac{a_1}{3} \left(\frac{\partial U}{\partial a} \right)_1 - U_1 \left\{ 1 - \frac{\rho_1}{\rho_m} \right\}. \qquad (12)$$

Since U is a positive function of a regardless of the density distribution, it follows that the superficial oblateness of a configuration rotating slowly with a given angular velocity at the surface will be greater if ω increases inwards than if it is constant throughout.

Conclusions.—Before applying results of the foregoing analysis to astrophysical cases we should bear in mind that stars are in radiative, not hydrostatic, equilibrium; and therefore equation (9) is not necessarily sufficient for specifying variation of ω. It is, for instance, possible that if the pressure is constant in the envelope of a star built up according to the Roche model, the braking action of radiation, discussed by Jeans,[7] may slow up the rotation of the outer layers and so cause the angular velocity to increase inwards. But, provided that

$$\lim a^5 \omega^2 = 0, \quad \text{if } a \to 0,$$

the external form of such a star remains unaffected—at least as far as terms of the second and higher orders in v can be neglected.

If we come to consider stars in whose interiors the pressure increases from the surface to the center, no rigorous reason is known which would compel such stars to rotate with uniform angular velocity. The possible variation of ω in stellar interiors has become a subject of numerous investigations in recent years; but despite considerable amount of work, the problem is still far from being clarified. The adoption of a definite law of the form $\omega^2(a)$ thus, at the present stage, appears premature. We may, however, recall that the external form of uniformly rotating configurations with pronounced density condensations comes very close to that corresponding to Roche's model. But the form of the latter was found to be independent of eventual variation of ω in the interior; so if

the analogy can be any guide, the form of slowly rotating real stars should be also almost completely specified by the surface values of ω and $\bar{\rho}$.

In addition, an important empirical indication in favor of the view that ω does not change rapidly in the "far interiors" of genuine stars has recently resulted from the study of the ellipticity effect in close eclipsing binaries. As is well known, the surface brightness of distorted stars depends on the local temperature gradient beneath the photosphere ("gravity effect"); and the latter, in turn, depends on the variation of ω in the interior. In the case of tidal distortion the consequences of the actual distribution of brightness are observable and can be compared with theoretical predictions. An exhaustive investigation of this effect has recently been carried out by the writer;[8] the result was that the intensity distribution observed for a number of representative Main Sequence stars came out remarkably close (with the uncertainty not exceeding 3 per cent) to that predicted on the assumption of uniform rotation. This can be taken as an additional indication in support of our conclusion that the external form of slowly rotating stars, possessing pronounced density condensations, is practically specified by their angular velocity of rotation at the surface and their mean density—irrespective of possible variation of ω in the interior.

* The existence of an absolute upper limit for the oblateness can also be inferred by a simple extension of Clairaut's theorem (cf. Tisserand, *Traité*, Tome II, p. 201) which states that the gravity g varies over the surface of a slowly rotating spheroid as

$$\frac{g - g_1}{g_1} = \frac{\Delta_2 - 5}{\Delta_2} \frac{r - a_1}{a_1},$$

where g_1 is the gravity at the surface of a sphere of radius a_1 having the same volume as the spheroid, and Δ_2 is identical with the expression in curly brackets on the right-hand side of equation (6.1). If the whole mass of a configuration is confined to an infinitesimally thin surface shell, the gravity should be independent of direction. A glance at the above equation shows that this is the case if $\Delta_2 = 5$. For homogeneous and centrally condensed models—as well-known—$\Delta_2 = 2.5$ and 1, respectively.

[1] *Théorie de la Figure de la Terre*, Paris, 1743. For its more detailed derivation cf. also Tisserand, *Traité de la Mécanique Céleste*, Paris, 1891, Tome II, pp. 211–213; or Lichtenstein, *Gleichgewichtsfiguren rotierender Flüssigkeiten*, Berlin, 1933, pp. 123–125.

[2] *Recherches sur la Figure des Planètes*, Paris, 1793.

[3] *Mem. Acad. Sci. St. Pétersbourg*, **15**, No. 10, p. 54 (1904).

[4] Cf. Tisserand, op. cit., p. 217.

[5] Cf. Kopal, *M. N.*, **98**, 414 (1938).

[6] Cf. Dive, Thèse, Paris, 1930, p. 24.

[7] *M. N.*, **86**, 444 (1926); or *Astronomy and Cosmogony*, Cambridge, 1928, pp. 266–271. *Ann. New York Acad. Sci.*, **41**, 13 (1941) (cf. pp. 43–44).

REMARKS ON THE PEGASUS CLUSTER OF NEBULAE

By J. B. Edson and F. Zwicky

California Institute of Technology, Pasadena

Communicated July 7, 1941

In continuation of some researches previously reported on the structure of clusters of nebulae we here discuss some of the data obtained with the 18-inch Schmidt telescope on Palomar Mountain concerning the Pegasus

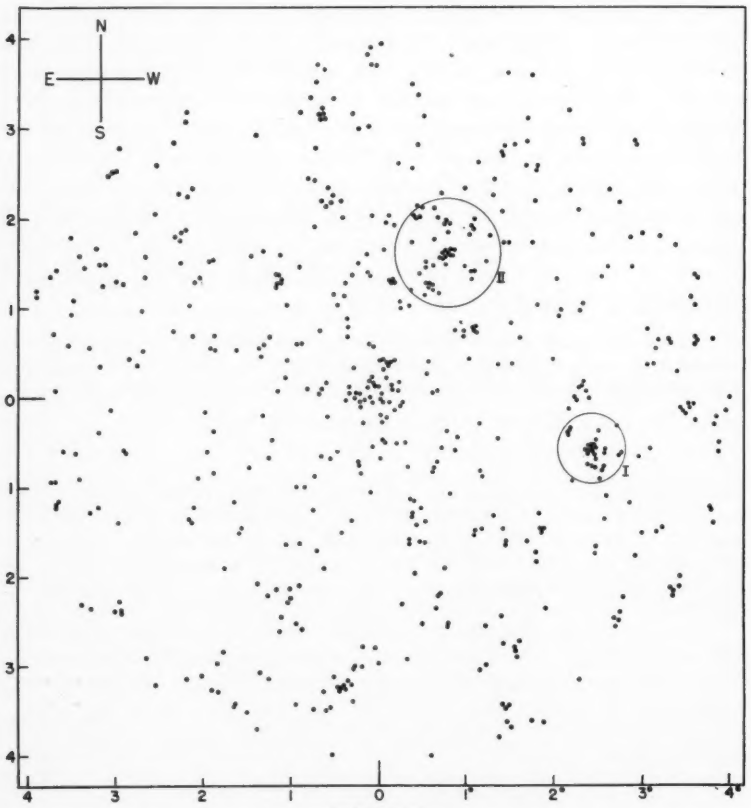


FIGURE 1

Distribution of nebulae around the Pegasus Cluster.

cluster of nebulae (R.A. $23^h\ 17^m$; Decl. $+ 7° 50'$, 1930; gal. long. 55°, lat. $-49°$). According to Hubble and Humason[1] this cluster consists of about 100 nebulae scattered over an area roughly 1° in diameter. The Pegasus

cluster lies at a distance of about 7.25 million parsecs and is centered around the two brightest objects NGC 7619 and 7626 which are slightly fainter than the photographic apparent magnitude 13. The most frequent apparent magnitude in the cluster is about 15.5.

The distribution of the nebulae brighter than about $m = 16.6$ in the cluster and in its immediate neighborhood is shown in figure 1 in which 556 nebulae are plotted. It is at once apparent that the field shown around the Pegasus cluster proper, which is located in the center, contains at least

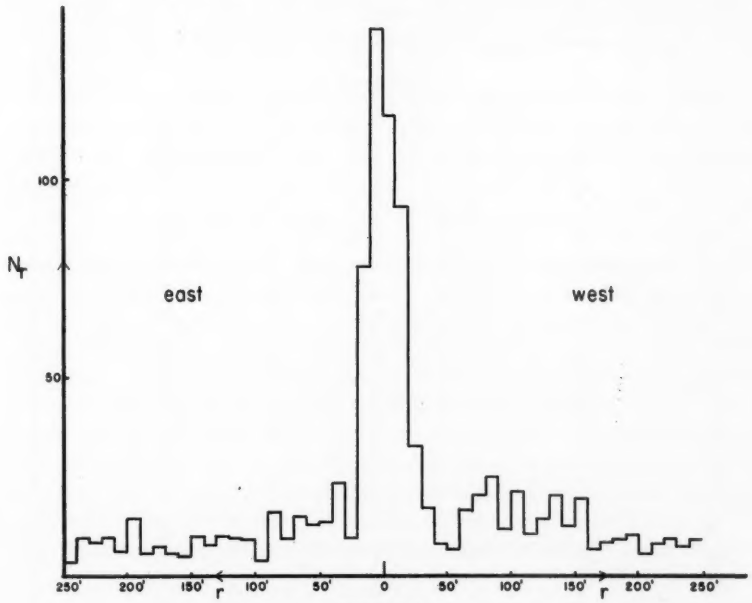


FIGURE 2
Radial distribution of nebulae in the Pegasus Cluster.

two other large groups marked by the two circles. The cluster enclosed by the circle I which is located in the constellation of Pisces was previously discovered by Zwicky. A plate taken with the 100-inch telescope and kindly put at our disposal by Dr. Baade shows this cluster to be of a compact type which contains at least 200 nebulae. Some of the structural characteristics of this interesting cluster will be discussed in another paper.

The field shown in figure 1 seems to form the southwestern end of the enormous cloud of nebulae extending from the constellation of Andromeda through Pisces to the large cluster of nebulae in Perseus. This cloud exhibits a characteristic mottled appearance which is due to the existence of

many relatively compact subgroups. Some of these subgroups in Pisces were previously discussed.[2]

Adopting as the center of the Pegasus cluster the point halfway between its two brightest objects the field was subdivided into rings with successive outer radii $r_n = n \times 10'$ arc. The number of nebulae in each ring was counted in the quadrants defined by the north south and east west directions, respectively. The results obtained are tabulated in table 1 and illustrated in figure 2.

TABLE 1

COUNTS OF NEBULAE IN RINGS WHOSE WIDTH IS 10 MINUTES OF ARC

RING	QUADRANTS SW	SE	NE	NW	n_r	N_r/SQ. DEGREE
1	3	3	3	2	11	127.6
2	6	4	6	6	22	85.0
3	1	1	1	6	9	20.9
4	3	2	5	2	12	19.9
5	1	3	2	2	8	10.3
6	3	3	3	0	9	9.5
7	5	2	6	4	17	15.1
8	3	3	3	10	19	14.7
9	6	3	8	12	29	19.8
10	4	1	2	5	12	7.3
11	3	0	8	16	27	14.9
12	4	5	4	6	19	9.6
13	3	5	5	12	25	11.6
14	13	3	6	10	32	13.7
15	8	5	7	7	27	10.8
16	18	5	2	7	32	11.9
17	8	5	3	1	17	6.0
18	8	3	8	4	23	7.6
19	6	4	5	8	23	7.2
20	6	10	14	10	40	11.9
21	3	6	5	6	20	5.7
22	9	10	8	5	32	8.6
23	8	8	8	9	33	8.5
24	6	10	9	8	33	8.1
25	15	4	3	3	25	5.9
Total	153	108	134	161	556	

The number of nebulae in the ring of outer radius r is n_r while the average number of nebulae per square degree in each ring is N_r.

The average number of nebulae per square degree as plotted in figure 2 was calculated separately for the half spaces east and west of the center of the Pegasus cluster. This brings into evidence the fact that the values of N_r in the western half space are relatively large for those rings which cover the two additional clusters.

It follows from figure 2 that the Pegasus Cluster is considerably smaller than the Hydra cluster[3] which lies at about the same distance. As stated by Hubble and Humason[1] the apparent diameter of the Pegasus Cluster is of the order of about 1°. Its intrinsic characteristics are probably very similar to some of the less populated more nearby groups of nebulae such as the group in Fornax (R.A. $3^h\ 30^m$; Decl. $-36°$) which we shall discuss in another place.

The general background of the field shown contains on the average about 8.5 nebulae per square degree. Of the 556 nebulae shown about 462 constitute the general background of the circular field of 250′ arc radius. The remaining 94 nebulae belong to the Pegasus Cluster (about 30) and to the two other clusters.

Our counts cover all nebulae which on the average are brighter than the apparent magnitude $m = 16.6$. According to Hubble[4] the number N_m of nebulae per square degree which are brighter than the apparent magnitude m is given by

$$\log_{10} N_m = 0.6m - 9.1 + 0.15(1 - \operatorname{cosec} \beta)$$

where the last term is the correction for average local obscuration at the galactic latitude β. For $m = 16.6$ and $\beta = -49°$ we therefore have $N_m = 6.5$ nebulae per square degree. This number is somewhat smaller than the average of 8.5 nebulae found in the background of our field, a result which is probably due to the fact that some nebulae of this background belong to the very extended cloud of nebulae mentioned previously.

On our films there is a marked scarcity in the number of nebulae as well as that of stars in a ring about 40′ from the center of the Pegasus Cluster. Whether this scarcity is accidental or whether it is due to some interstellar obscuration cannot here be decided, although it seems unlikely that at the galactic latitude of $-50°$ much obscuring material should be expected.

[1] E. Hubble and M. L. Humason, *Astrophys. Jour.*, **74**, 131 (1931).
[2] F. Zwicky, these PROCEEDINGS, **23**, 251 (1937).
[3] F. Zwicky, *Ibid.*, **27**, 264 (1941).
[4] E. Hubble, *The Realm of Nebulae* (Yale University Press, 1936, p. 179).

THE INACTIVATION OF BACTERIOPHAGES BY X-RAYS—INFLUENCE OF THE MEDIUM

BY S. E. LURIA AND FRANK M. EXNER

FROM THE BACTERIOLOGICAL RESEARCH LABORATORIES OF THE DEPARTMENT OF SURGERY, AND FROM THE DEPARTMENT OF CANCER RESEARCH, COLLEGE OF PHYSICIANS AND SURGEONS, COLUMBIA UNIVERSITY, NEW YORK, N. Y.

Communicated July 12, 1941

Several studies on the inactivation of bacteriophages by x-rays have recently been published.[1] Qualitative experiments of Wollman and Lacassagne have shown that the x-ray sensitivity of different phages suspended in broth increases with their particle size as determined by ultrafiltration and centrifugation.[2] Holweck, Luria and Wollman have analyzed the inactivation of C16 phage by x-rays and alpha-particles, and interpreted their results in terms of the "hit theory." Further discussion of the same data has been given by Lea.[3]

We have carried out experiments with x-rays on a series of bacteriophages (3K, C16, C13, P28), some of which have been measured by Elford.[2] In all experiments the concentration of phage particles was determined by the plaque count method. Samples were irradiated in celluloid tubes with careful attention to x-ray dosage measurement. The same x-ray generator was used for all exposures. Technical details will be described in a complete account to be published later.

1. When bacteriophages are irradiated in the original filtrate or in broth dilutions the results (Figs. 1 and 2) can be summarized as follows:

(*a*) The curve of inactivation (percentage of remaining phage as a function of the dose of x-rays measured in *r*) is exponential in the first part of its path (straight line in semi-logarithmic plot). When the dosage becomes very high (Fig. 2) the rate of inactivation shows an increase, which however, is so slight as to be detectable only because of the high precision obtained in the phage assay. This increase in inactivation rate with high dosage had also been found in the experiments of Exner and Zaytzeff-Jern[1] and is probably due to superposition on the principal inactivation process of some effect of a different kind. We hope to determine the nature of this effect.

We have followed the inactivation to a diminution in titer by a factor of 2×10^{-7} and have always found consistently reproducible results. The inactivation is a function of the total dose of x-rays, and not of the dosage rate (no "time factor"). The fractional inactivation produced by a given dose is independent of the concentration of the phage suspension during irradiation.

(*b*) For x-ray tube voltages between 200 and 1000 kv. the inactivation rate has been found independent of the quantum energy (wave-length) of the x-rays. This statement refers to x-ray dosage measurements based on a Victoreen thimble chamber (wall thickness increased for measurements at highest voltage). It is of interest to point out the difference between this result and that obtained with tobacco mosaic virus by Gowen,[4] who finds a definite wave-length effect.

(*c*) The x-ray sensitivity of each bacteriophage is a highly reproducible property of that strain. The inactivation curve for phage C16 obtained a year ago by Holweck, Luria and Wollman in Paris coincides exactly with the curve established by us on the same phage. The two samples, though of common origin, had been kept many years in different surroundings and grown on bacterial hosts of different species (Dysentery Y6R and Coli 234).

(*d*) The slope of the inactivation curve is different for the different phages (Fig. 1). Phages of smaller particle size are more resistant. In table 1 are shown the dimensions of the "sensitive volumes" as calculated

TABLE 1

BACTERIOPHAGE	SUSCEPTIBLE ORGANISM	INACTIVATION DOSE* IN ROENTGENS	TARGET DIAMETER, MILLIMICRONS	PARTICLE SIZE (ELFORD) MILLIMICRONS
3K	Staphylococcus "K"	45,000	48	50–70
C16	Coli 234 or Dysentery Y6R	40,000	50	50–75
C13	Coli 234	115,000	32	25–30
P28	Coli "PC"	90,000	36	.....

* Dose giving inactivation ratio $N/N_o = 1/e$.

according to Lea's method,[3] and the size of the particles as given by Elford.[2] It is evident that the two series of values fall close to one another, in agreement with the hypothesis (Holweck, Luria and Wollman) that in the case of bacteriophages the sensitive volume corresponds to the phage particle itself. The small particle size of P28 phage seems to be confirmed by centrifugation experiments now in progress. The utility of the radiation method as a means of gaining information about the size of ultramicroscopic structures seems to be well borne out by these results.

2. We wish now to call attention to a new aspect of the inactivation of bacteriophages by x-rays, which becomes evident when phages are suspended in different media.

If a bacteriophage filtrate is diluted in distilled water, instead of broth, so that less than 1 per cent of the original filtrate is present, *its inactivation by x-rays is found to be greatly increased.* A dose of x-rays that leaves 40 per cent active phage in broth inactivates more than 99 per cent of phage diluted 10^{-3} in distilled water. This effect and the other facts related below have been found to hold for all four phages so far investigated, although

the most thorough of our studies have been made on the 3K and C16 phages. We have established the following facts:

(*a*) The effect of distilled water as a medium does not depend on difference of pH as compared to broth. Broth and water suspensions with pH adjusted to different values between 6.8 and 8.0 were irradiated. All broth suspensions gave closely the same inactivation rate. On the other hand, the water suspensions all showed the higher order of sensitivity found with distilled water.

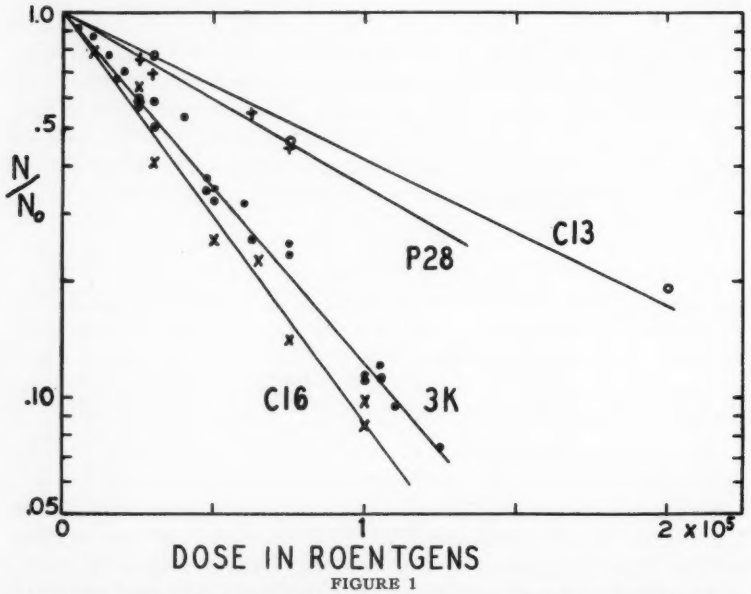


FIGURE 1

Inactivation of four bacteriophages suspended in broth. Abscissas: dose of x-rays. Ordinates: proportion of phage particles remaining active after irradiation.

(*b*) Suspensions of phage in different saline solutions (Na, K, Ca and Mg in various proportions) and in phosphate buffer show sensitivities of the same order as that of water suspensions.

(*c*) Suspensions of phage in aqueous solutions of ordinary gelatin or of Eastman Purified Gelatin behave like broth suspensions whenever gelatin is present in concentrations above 10^{-4} gm. per cm.[3]

(*d*) Protection against x-ray inactivation, similar to that exerted by gelatin and broth, though less complete, was found when solutions of egg albumin and serum albumin were used as media.

(*e*) No protection was given by either reducing or non-reducing sugars (dextrose, sucrose).

(*f*) Oxidizing (H_2O_2) and reducing agents (Sodium Thioglycolate and Cysteine Hydrochloride) added to un-irradiated phage suspensions inactivated the phage. Similar observations have been reported by other

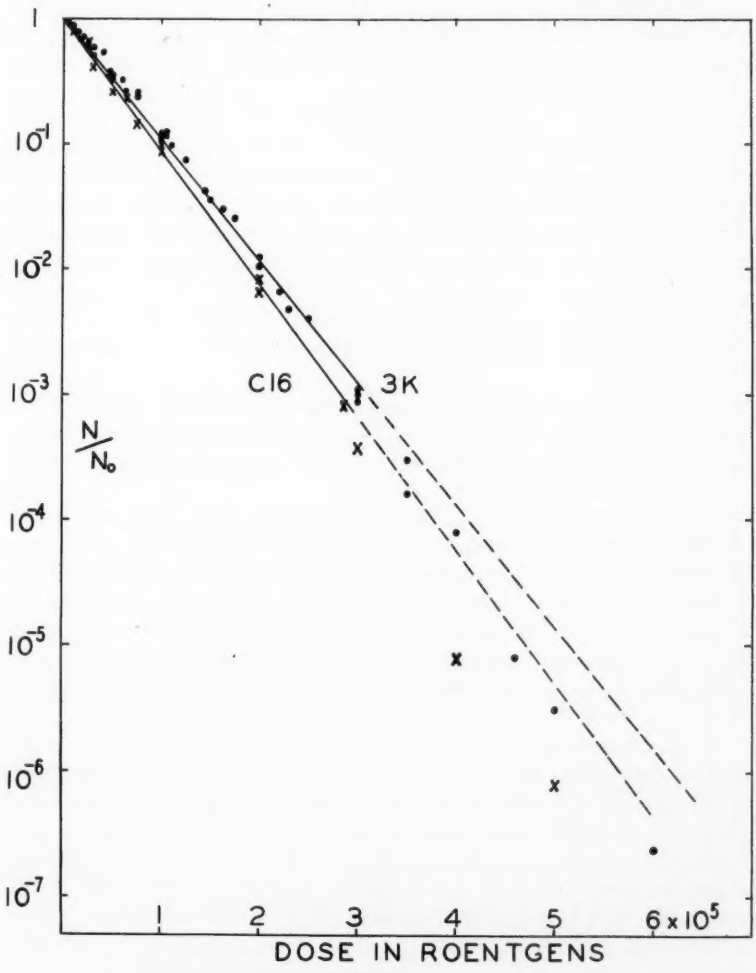


FIGURE 2

Extension of two of the inactivation curves shown in figure 1.

workers.[5] We have found such inactivation to be much more rapid in distilled water, saline, etc., than in those media (broth, gelatin, etc.) which protect phages against the action of x-rays. Concentrations of protective

substance necessary to give effective protection are the same in both cases. On the other hand concentrations of thioglycolate and cysteine low enough to have slight or no effect on the activity of phage gave no protection against x-ray inactivation.

(*g*) Broth, gelatin and albumins give complete protection against the slow spontaneous inactivation which phages undergo when highly diluted in distilled water or saline solution.

(*h*) With progressively lower dilution in distilled water, so that the concentration of original broth filtrate is progressively higher, the inactivation rate of phage by x-rays diminishes. At concentrations of broth above 1 per cent the sensitivity to x-rays becomes constant, and identical with that of suspensions in undiluted broth. The protection is not due to substance of bacterial origin present in the filtrate since the addition of 1 per cent of new broth or of as little as $1/10^4$ gm. per cm.^{3} gelatin to dilute suspensions of phage in water is enough to protect phage in the same measure. These quantities of broth and gelatin represent roughly comparable amounts of protein.

(*i*) No amount of gelatin or albumin added to the broth suspensions of phage gives any additional protection. *There exists a minimum sensitivity to x-rays for each phage*, which is the sensitivity measured in broth suspensions and analyzed in part 1 above.

(*j*) The action of x-rays on phage suspensions in water is not, as in broth, a function of the dose of radiation only. A large "time factor" is found, in the sense that a given dose is more effective the longer the time over which it is spread (within the range tested). We have not yet been able to establish whether inactivation continues after the end of the irradiation. However, water irradiated alone and inoculated immediately thereafter with phage failed to show higher inactivating power than un-irradiated water.

(*k*) The inactivation by x-rays of *different phages* suspended in water does not follow the same rules as the inactivation in broth. First it should be said that it is difficult to establish the shape of the inactivation curves owing to the presence of spontaneous inactivation and to the "time factor." However, the order of difference in sensitivity between small and large phages is clearly reversed: small phages suspended in water are inactivated somewhat more easily than large ones. In this connection it is of interest that these small phages are more sensitive also to the action of heat.[6]

Discussion.—The observations listed under parts 1 and 2 above appear as a whole to fit the following interpretation:

The inactivation of bacteriophages by x-rays is at least a twofold effect.

First there is an "indirect" effect predominant in water or saline suspensions, as described under part 2 above. This effect is probably exerted

through some inactivating agent produced in the medium as a consequence of the irradiation. This agent has a short life, as no inactivating effect is shown by previously irradiated water.

The mode of action of this agent on phages resembles that of oxidizing and reducing agents in that protection is afforded in both cases by the presence of small quantities of foreign proteins. This protection seems likely to be due to competition between the foreign protein and phage in reacting with the inactivating agent.

In the presence of foreign proteins bacteriophages are still inactivated by x-rays, but with reduced sensitivity and with other relationships as listed under part 1 above. This action we consider to be a "direct" effect on the phage particle itself, which can be analyzed according to the pattern of the "hit theory." This treatment will be developed more fully in a subsequent paper.

The indirect effect of x-rays on bacteriophages through the medium appears similar to the effects obtained by different authors[7] with enzymes and simple proteins.

A protective action of foreign proteins against x-ray inactivation of papilloma virus suspended in saline has recently been observed.[8] This is apparently another instance of the type of protection described above for phage.

It has been shown that with bacteriophages under suitable conditions both a direct and an indirect effect of radiation can be distinguished. This should serve to emphasize the importance of considering both types of action when analyzing the effect of radiation on a particular biological system.

We are indebted to Dr. G. P. Berry, Dr. J. Bronfenbrenner and Dr. A. P. Krueger for kindly supplying the strains of bacteriophages and bacteria used in these experiments.

[1] Wollman, E., and Lacassagne, A., *Ann. Inst. Past.*, **64**, 5 (1940).
Holweck, F., Luria, S., and Wollman, E., *Compt. rend. Acad. Sci*, **210**, 639 (1940).
Wollman, E., Holweck, F., and Luria, S., *Nature*, **145**, 935 (1940).
Exner, F. M., and Zaytzeff-Jern, H., *J. Appl. Phys.*, **12**, 338 (1941).

[2] Elford, W. J., in Doerr and Hallauer, *Handbuch der Virusforschung*, page 126, Julius Springer, Wien, 1938.

[3] Lea, D. E., *Nature*, **146**, 137 (1940).

[4] Gowen, J. W., *Proc. Nat. Acad. Sci.*, **26**, 8 (1940), and personal communication.

[5] Lominski, I., *Compt. rend. Soc. Biol.*, **119**, 952 (1935), and **122**, 766 (1936).

[6] Luria, S. E. (unpublished results).

[7] Fricke, H., *Cold Spring Harbor Symposia on Quantitative Biology*, **2**, 241 (1934).
Dale, W. M., *Biochem. J.*, **34**, 1367 (1940).

[8] Friedewald, W. F., and Anderson, R. S., *Proc. Soc. Exptl. Biol. Med.*, **45**, 713 (1940).

BINGHAMIA, THE ALGA, VERSUS BINGHAMIA, THE CACTUS

BY WILLIAM ALBERT SETCHELL AND ELMER YALE DAWSON

DEPARTMENT OF BOTANY, UNIVERSITY OF CALIFORNIA

Communicated July 7, 1941

J. G. Agardh, in 1894, in the Continuatio II of his *Analecta Algologica*, made an observation at the bottom of page 63 which may be roughly translated as follows: "I am willing to constitute another genus, intermediate between *Herpophyllum* and *Nitophyllum*, given to me by Farlow from California under the name, unless I mistake myself, *Binghamiae*, by an author unknown to me. This plant, plane, strictly dichotomous, but augmented by segments arising from the margins, shows the habit of *Nitophyllum*, but has an entirely different structure in that internal and irregular lacunae are present between the surface tissues, the intervening diaphragms being rather irregularly cellular and at length sporangiiferous. I have seen cystocarps agreeing with those of other Delesseriaceae and the tetrasporangia grouped into sori, but from the structure of the frond I assume this plant to present a type diverse from that of other Delesseriaceae. I am ignorant as to where this plant was described, nor do I know whether a lawful name was bestowed upon it." So far as is known to us, this is the first time the name *Binghamia* appeared in print. Here it is accompanied by a partial diagnosis, recognized as of a distinct genus, but with an indefinite sponsor. No type species is mentioned, and the name is not in the nominative, but in the genitive form, "Binghamiae."

In August, 1892, J. G. Agardh, in a letter to W. G. Farlow wrote thus: "11.—The plant from California about which you asked me if it would be a new genus of mine, named *Binghamia*, seems really to be a new genus; but I don't remember that I have ever seen the plant before. Nor, I think, have I given that name to some other plant, though certainly I should wish that a genus of California remembered of the discoverer of several new California plants." This passage from J. G. Agardh's letter explains his mystification as expressed in various publications.

Farlow, too, was equally "mystified," since, from Agardh's letter he seems to have forgotten an earlier letter, to Mrs. C. P. Bingham, at that time in Santa Barbara, California, dated February 11 (probably 1879) in which he says: "The other plant first received from you and supposed to be a *Rhodophyllis* near *R. bifida* is undoubtedly a new species and perhaps a new genus. It is not a *Rhodophyllis* as the tetraspores are not zonate. Provisionally we will call it *Binghamia californica*, expecting, however, that farther study will induce us to put it into some old genus, in which case the species will be ———— *Binghamii*." It seems, therefore, that Farlow is responsible for the binomial *Binghamia californica* Farlow *in*

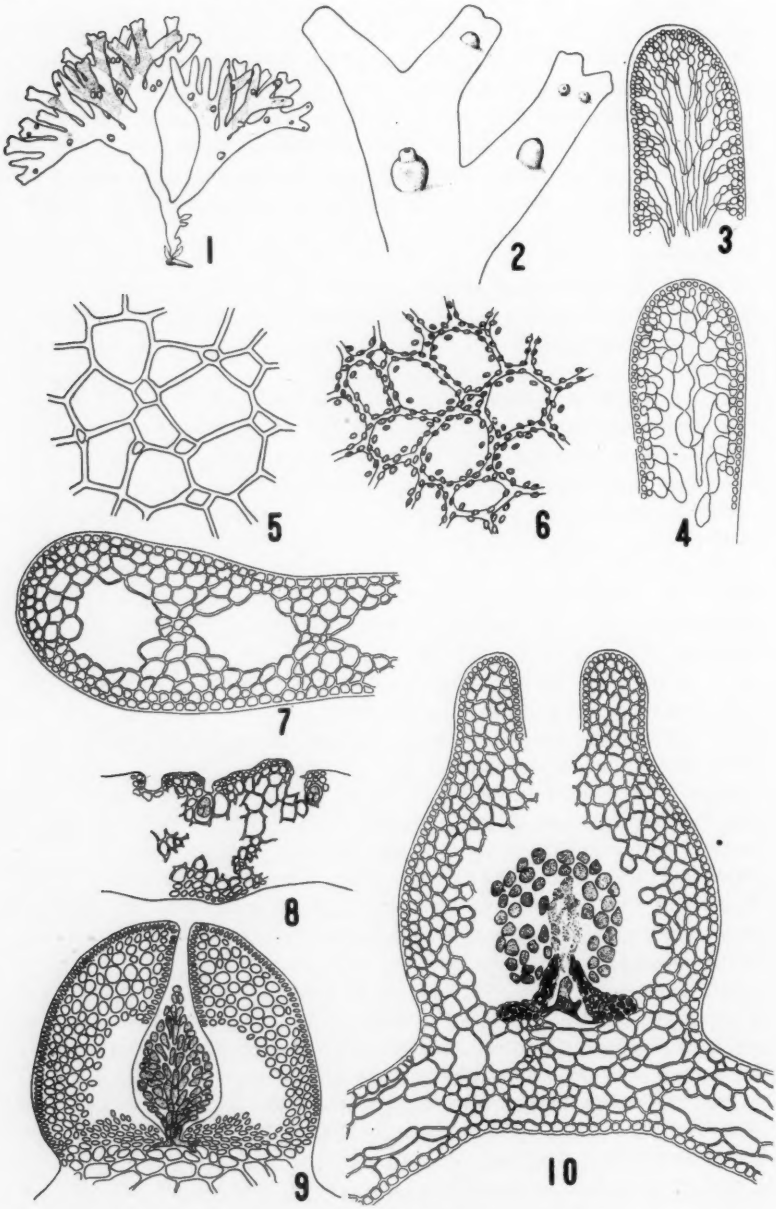
lit., but forgot this earlier christening of the species when he wrote to Agardh in 1892. In reply to his letter, Mrs. Bingham, on April 24 (1879) merely acknowledges the naming of the alga after her.

J. G. Agardh, evidently, kept the matter in mind, for in 1898, in his *De Dispositione Delesseriearum* under *Herpophyllum australe* J. Ag., he again speaks of having another species, sent to him under the name *Binghamia* and long preserved in his herbarium under this name, whether proposed by himself or by another, he does not at that time dare to say.

In 1899, J. G. Agardh, in his *Analecta Algologica*, Continuatio V, provides figures of portions of both the cystocarpic and the tetrasporic plants of *Binghamia californica*, with analyses of slight detail in the explanation. The figures, however, do not seem sufficiently accurate to distinguish the species.

In recent Japanese phycological literature the name *Binghamia californica* Farlow appears several times. At least three authors have credited the species as occurring in Japan, and a comparison of specimens from California and Japan reveals little upon which to base a firm distinction. The wide geographical discontinuity, however, leads us to suspect that distinction may ultimately be made. It was first identified and illustrated by Inagaki in a paper written in Japanese (1933). Segawa (1936 and 1938) records the species from the province of Izu and in the later publication illustrates tetrasporic sori. Okamura (1936, in Japanese) describes and illustrates *Binghamia californica*. He had apparently made some investigation of the history of the nomenclature, for he gives the name as *Binghamia californica* Farlow mscr.

The genus *Binghamia* of red algae might be thoroughly validated by our giving complete Latin diagnoses of both the genus and the species *B. californica*, and continued without further discussion, were it not for the fact that Britton and Rose (*Cactaceae*, **2**, 167 (1920) and **4**, 279 (1923) proposed a genus *Binghamia* with four species, later increased to 19 by Backeberg, Werdermann and West. Most, if not all, of the species of this South American Cactaceous genus are in cultivation and some have become very well known. Though the algal binomial *Binghamia californica* may be considered somewhat better than a "nomen nudum," yet, on three points it is not acceptable according to the International Rules of Botanical Nomenclature, 1935: (1) The description by J. G. Agardh in 1894 was not in the form of a diagnosis, and no species was mentioned as a basis for the genus. (2) When the binomial *Binghamia californica* was first suggested by Farlow in the letter to Mrs. Bingham, it was used only as a tentative name. (3) The figures published by Agardh in 1899, accompanied only by meager diagnosis in the explanation of plates, do not show sufficient essential characters to be accepted under specification 3 of Article 44. Since the algal genus *Binghamia* cannot be considered



SEE OPPOSITE PAGE FOR EXPLANATION OF PLATE

validly published, there are no rules which bar it from use for the genus of cacti. In order that the algal genus may fulfil the intentions of Farlow and J. G. Agardh as expressed earlier in this paper, and may maintain its alphabetical order in indices, it seems allowable to suggest the name *Binghamiella* in honor of Mrs. C. P. Bingham, leaving the Cactaceous genus to bear the name *Binghamia* in honor of Dr. Hiram Bingham, to whom it was dedicated and validated without question.

Binghamiella gen. nov.

Binghamia Farlow *in lit.*, 1879; J. G. Agardh, Analecta Algologica, Cont. II; 63, 1894 (footnote), Sp. Alg., 3 (3); 140, 1898, Sp. Alg. 3 (4); 134, 1901; Okamura, Descriptions of Japanese Algae (in Japanese); 689, 1936; Segawa, in Sci. papers of the Inst. of Alg. Research, Fac. of Sci., Hokkaido Imp. Univ., II, 1; 150, 1938.

Frons e disco explanata pauca, compressa, sursum in laminis planis fere regulariter dichotomis; segmentis adultis anguste elongato cuneata intus cellulis magnis lacunas parvas irregulares medias includentibus, extus cellulis corticalibus paucistratosis, majoribus aliquante superficie hexagoniis et quibusque majoribus cellulis multo parvioribus imperfecte cinctis, iis apicalibus (et in partibus interne meristematicis) elongatis brevi filamentoideis et in modo fontis exsilientis ordinatis; tetrasporangiis tripartitis, in cellulis externis oriendis, aetate provecta in soris, irregularibus, in cavernis vadosis

---

EXPLANATION OF PLATE

Binghamiella californica Setchell & Dawson

1. Habit of cystocarpic plant. × 1 diam.
2. Upper lobes of same plant showing cystocarps in different stages of development. × 4 diam.
3. Longitudinal section through a young tip of a frond, showing fountain type of apical growth. × 58 diam.
4. Longitudinal section of older tip, showing condensation of filaments and beginnings of lacunae. × 58 diam.
5. Surface view of adult frond of a cystocarpic plant. × 225 diam.
6. Surface view of adult frond of a tetrasporic plant, the small, shaded cells projecting slightly beyond the ordinary cells. × 225 diam.
7. Transverse section of an adult, cystocarpic frond showing lacunae, dissepiments and cortical tissues. × 58 diam.
8. Transverse section of a tetrasporic plant (typel) showing tripartite tetraspores in sunken pits. × 50 diam.
9. Vertical section through a young cystocarp, showing young, thick pericarp and young carpostome yet without "collar-like" rostrum, the pulvinus (or cushion) at the base, and the young gonimolobes. × 100 diam.
10. Vertical section through a mature cystocarp with portions of adjacent frond, showing lacunae and adjacent tissues in the frond, the cushion, and ripe carpospores surrounding the disintegrating sterile portions of the gonimolobes (a composite drawing). × 58 diam.

sparsis et superficie per magnitudine conspicuis; cystocarpiis sparsis nunc singulis, nunc pauciaggregatis, sessilibus, infra semiglobosis, superne abrupte angustatis brevicylindricisque, et per ostiolum late obconicum perforatis; gonimolobis massam globosam formantibus, inferne brevi stipitatis, superne in sporis plerumque transformatis; carposporis pyriformibus in modo cystocarpii Rhodymeniearum se ostendentibus; antheridiis nondum visis.

Binghamiella californica (Farlow *in lit.*) comb. nov.

Binghamia californica Farlow *in lit.* (1879); J. G. Agardh, Anal. Alg., Cont. V; 158, pl. 1, fig. 6a, 6b, 1899; Daniel Cleveland, Mar. Alg. of San Diego, 1880; Inagaki, Oshorowan oyobi soreni kinsetu seru Engan no Kaisankosōrui (Hōkkaidō-Teikoku-Daigaku, Rigakubu, Kaisō-kenkyuzyo Hokuku, No. 2) (in Japanese); 43, f. 16, a-c, f. 17, a-b; Segawa, Mar. Alg. of Susaki, in Sci. papers of Inst. of Alg. Research, Fac. of Sci., Hokkaido Imp. Univ., I, 2; 188, 1936, II, 1; 149, 1938; Okamura, Descriptions of Japanese Algae (in Japanese) 689, 1936.

Frons rosea, usque ad 6 cm. alta et plus minusve 6–8 dichotomis; segmentis divaricatis, basi leviter attenuatis, margine integra aut laminas parvas lobulaque emettente, usque ad 2–3 mm. latis et plus minusve 300 μ crassis, apice attenuato obtusis aut bilobato late rotundatis; tetrasoris numerosis, plus minusve delimitatis, irregulariter rotundatis per superficiem totam profuse sparsis in cavernis, parvis et angustis, circa 65 μ in diam. et vert. et horiz.; cystocarpiis singulis aut pauciaggregatis; antheridiis nondum visis.

Type specimens: tetrasporic, Herb. Univ. Calif. 93480, Santa Barbara (ex Herb. Lorenzo G. Yates); cystocarpic, Herb. Univ. Calif. 266343, Santa Barbara.

The investigations of the Japanese workers have placed this genus and species in the family Champiaceae of the Rhodymeniales. Okamura has placed it in subfamily Champiae, Segawa, in subfamily Lomentarieae. A study of the tetrasporic plant shows that it belongs in the Lomentarieae as limited by Kylin (1931) since it has the tripartite tetrasporangia borne in shallow cavities as in *Lomentaria*. The cystocarps, too, agree with *Lomentaria*. It differs from *Lomentaria* in the structure of the thallus, being lacunose within instead of continuously tubular.

We wish to extend our sincere thanks to Professor David H. Linder, Curator of the Farlow Herbarium at Harvard University, for searching through the collections under his charge, and for sending us both Farlow's specimens and valuable extracts from his correspondence. To the Director and curators of the New York Botanical Garden we are indebted for the loan of specimens.

Agardh, J. G.
1894—*Analecta Algologica*, Continuatio II; 65.
1898—*Species, genera et ordines Algarum*, III, 3; 140.
1899—*Analecta Algologica*, Continuatio V; 158, pl. 1, figs. 6a, 6b.
1901—*Species, genera et ordines Algarum*, III, 4; 134.
Britton, N. L., and Rose, J. N.
1920—*Cactaceae*, 2; 167.
1923—*Cactaceae*, 4; 279.
Cleveland, Daniel
1880—*Marine Algae of San Diego.*
Inagaki
1933—*Oshorowan oyobi soreni kinsetu seru Engan no Kaisankosōrui (Hōkkaidō-Teikoku-Daigaku, Rigakubu, Kaisō-kenyuzyo Hokuku*, 2); 43, f. 16, a-c, f. 17, a-b (in Japanese).
Kylin, H.
1931—*Die Florideenordnung Rhodymeniales, in Lunds Universitets Årsskrift. N. F. Avd.* 2, 27, 11; 26.
Okamura, K.
1936—*Descriptions of Japanese Algae* (in Japanese); 689.
Segawa, S.
1936—*Marine Algae of Susaki*, in Sci. papers of the Inst. of Algol. Research, Fac. of Sci., Hokkaido Imperial University, I, 2; 188.
1938—*Ibid.*, II, 1; 149.

GROWTH REGULATORS OF PLANTS AND FORMATIVE EFFECTS INDUCED WITH β-NAPHTHOXY COMPOUNDS

BY P. W. ZIMMERMAN

BOYCE THOMPSON INSTITUTE FOR PLANT RESEARCH, INC.

Read before the Academy April 28, 1941

β-Naphthoxyacetic acid is a growth regulator with a unique capacity for inducing formative effects of leaves, stems, flowers and fruit. It has in common with other substances of the auxin group the power to cause cell elongation, increase cell division and induce adventitious roots. There are, however, many qualitative differences in the capacity of growth substances to induce physiological responses. Of the fifty or more known growth substances[9] the more active ones have some distinguishing characteristics when considered in relation to the various responses which they can effect.

β-Naphthoxyacetic acid was first mentioned as an active substance in 1938 by Irvine.[6] In 1939 it was cited with other naphthoxy compounds by Zimmerman and Hitchcock[9] as an active substance when applied in solution or as a vapor. Bausor[1,2] published two detailed reports in 1939 and in 1940 Bausor, Reinhart and Tice[3] reported on histological peculiarities induced with β-naphthoxyacetic acid. Killeffer[7] in 1940 published

a review of papers presented at a symposium of the American Chemical Society on plant hormones in September, 1939. The review contains graphic formulae of 19 active substances, four of which were β-naphthoxyacetic acid, β-naphthoxyacetamide, β-naphthoxypropionic acid and β-naphthoxybutyric acid.

The present report deals with the formative effects of β-naphthoxyacetic acid, its derivatives and homologs, and how the induced responses distinguish naphthoxy compounds from other growth regulators.

Materials and Methods.—The naphthoxy compounds used in the experiments were synthesized in the Institute laboratories.[10] In the results of experiments reported herewith, β-naphthoxyacetic acid was used in water solutions and emulsions while the methyl and ethyl esters were used when plants were exposed to vapors. β-Naphthoxypropionic acid and β-naphthoxyacetamide were used in water solution and lanolin emulsions. The most satisfactory emulsion is the one recommended by Carbide and Carbon Chemicals Corp. in a pamphlet called "Emulsions." The triethanolamine used in the emulsion is a very effective solvent for naphthoxy compounds. The effective concentrations ranged from 50 to 500 mg./l. and were varied according to the uses.

Plants used in the experiments were tomato (*Lycopersicon esculentum* Mill.), Turkish tobacco (*Nicotiana tabacum* L.), sensitive plant (*Mimosa pudica* L), fuchsia (*Fuchsia hybrida* Voss.), Jerusalem artichoke (*Helianthus tuberosus* L.), marigold (*Tagetes erecta* L.), foxglove (*Digitalis purpurea* L.), lily (*Lilium longiflorum* Thunb.), cucumber (*Cucumis sativus* L.), velvet tree (*Gynura aurantiaca* DC.), hydrangea (*Hydrangea opuloides* Koch), larkspur (*Delphinium ajacis* L.), squash (*Cucurbita maxima* Duchesne var. Hubbard), rose (hybrid teas), dahlia (*Dahlia variabilis* Desf.), coleus (*Coleus blumei* Benth.) and hibiscus (*Hibiscus rosa-sinensis* L.).

Results.—β-Naphthoxyacetic acid compared and contrasted with other growth substances. In the search for facts about growth and development of plants there have been located some 50 synthetic chemical compounds which induce hormone-like responses.[9] These are commonly referred to as growth substances, auxins, root-inducing substances, growth regulators and hormone-like compounds. None of them has been extracted from green tissue and chemically identified as a natural hormone. Some of them are known to exist in microörganisms, corn oil, woody tissue and urine. It is possible, however, to extract from green tissue unidentified substances with hormone activity. The more active synthetic compounds which act like hormones are as follows: unsaturated carbon gases (CO, C_2H_2, C_2H_4, C_3H_6), α-naphthalene, β-naphthoxy, β-indole, phenyl, anthracene, and fluorene acids, salts and esters. The amides of α-naphthaleneacetic acid and β-naphthoxyacetic acid are also active. Amides of some of the others have not yet been tested.

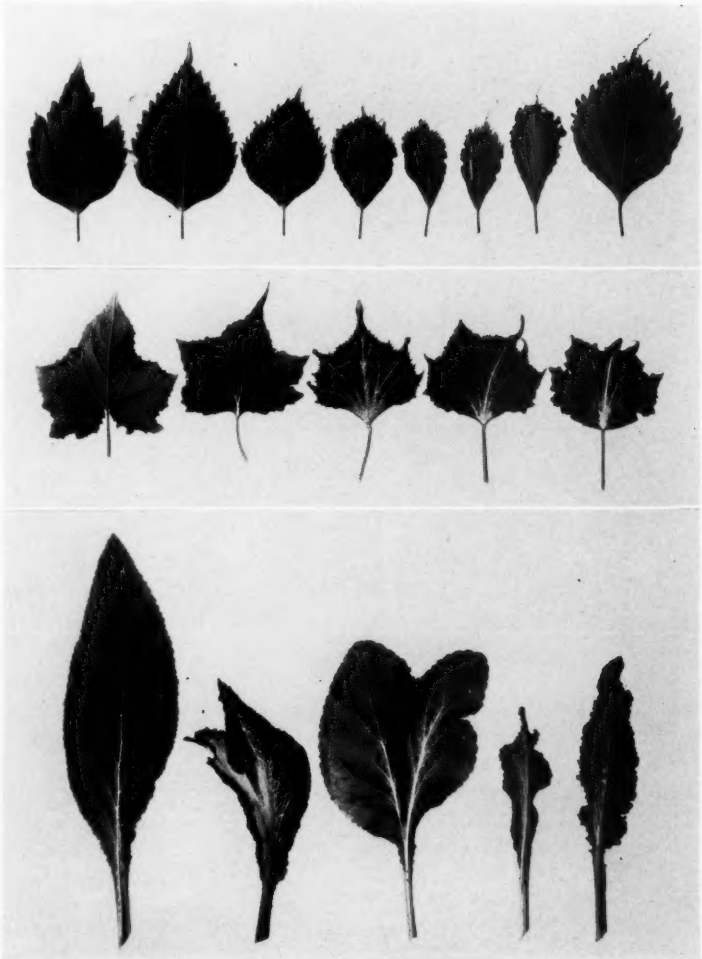


FIGURE 1

Leaves of three species to show modifications in size, shape and pattern, and clearing of veins induced by sprays containing β-naphthoxyacetic acid applied at the growing tip. Normal control leaf on left in each case. The modified leaves to the right grew after the tip was treated. Above, hibiscus; middle, cucumber; below, *Digitalis*.

FIGURE 2

Tomato to show induced parthenocarpy and modification of flowers, calyxes and entire fruit. Normal control on left in each case. *A*, flower clusters four days after treatment with 300 mg./l. *B*, flower buds modified from soil treatment with 10 mg. of β-naphthoxyacetic acid in 50 cc. of water. The tubular buds were sprayed with 300 mg./l. to induce parthenocarpy. *C*, fruit developed after treatment described in *B*. *D*, calyxes of modified fruit.

These synthetic growth regulators have in common the capacity to induce tropic responses (epinasty of leaves, curling and twisting of stems and leaves); abnormal growths (tumor and gall-like and overgrowths of callus); cause inhibition of growth in general and of axillary buds on growing stems or buds of tubers, corms, etc.; retard abscission of leaves, flowers and fruit; cause parthenocarpic development of ovaries; and initiate adventitious roots. These are all examples of growth regulation, though there are many qualitative differences in the responses induced by the different substances.

β-Naphthoxyacetic acid is especially noteworthy since, in addition to the capacity to induce the responses listed above, it also causes pronounced formative effects on leaves, tendrils, flowers and fruit. These responses are evident only on new growths which occur after the growing tips or entire plants are treated with the solutions, emulsions or vapors of β-naphthoxyacetic acid or its derivatives.

Modification of leaves and tendrils. The formative nature of β-naphthoxyacetic acid and its derivatives became evident when new leaves grew after the plants had been treated. The degree of modification varied with the species, the method of treating, the concentration of the chemical and the number of replicate treatments. The recovery of the plants also varied with the concentration. The most effective treatments were applications to soil of potted plants (15 to 50 cc. of 100 to 200 mg./l. to a four-inch pot), growing tip sprayed with water solution or emulsion containing 100 to 500 mg./l. of the chemical, lanolin preparation (0.5 to 1.0 per cent) applied near the growing tip, and exposure of the entire plant under a bell jar to vapors of the methyl or ethyl esters of β-naphthoxyacetic acid (the equivalent of one drop on a watch glass set on an inverted hot crucible).

Though there was considerable variation with different species, the new leaves produced by treated plants showed modification in size, shape, texture, number of leaflets on compound leaves, serrate edge, lobes, pattern of venation and transparency of veins. Some of these characteristics are evident in figure 1. The peculiar pattern and clearing of the veins are two of the most striking effects. When viewed with transmitted light the modified leaves appeared to have nearly transparent veins, a characteristic usually associated with virus diseases. The shape and texture of tomato and tobacco leaves resembled those of mite-infested plants.

The tendrils of control cucumbers were simple. Those of treated plants were frequently branched and leaf-like. The tendrils of squash, which are normally branched, often became simple or profusely branched. This method of inducing modification in structure might in time be of value in studying plant relationships.

Species which normally have simple leaves produced lobed leaves (Fig. 1) and compound leaf types sometimes produced simple leaves. With

high dosages sprayed on the tip the new leaves grew together, forming a circle around the growing tip. This type of response was common with coleus, Jerusalem artichoke, velvet tree, dahlia and sensitive plant. Fasciated leaves were induced on sensitive plant and tomato where leaflets failed to separate.

The most lasting effects came with soil treatments where 10 to 15 mg. of β-naphthoxyacetic acid were applied to a four-inch pot in 50 cc. of water. Without retreatment the tomato plants continued to produce modified leaves and flowers for 60 days or more.

Modified flower buds and fruit. When tomato plants were treated on the soil as described in the previous paragraph or when the stem was treated with a 1.0 per cent lanolin preparation, the flower buds were greatly modified. The calyx often formed a tube which prevented the flower from opening, though there were several variations of this. In a single cluster there were buds which split on one side, or in two or more places. The calyxes of ripe fruit shown in figure 2 (*B*) and (*D*) illustrate several variations.

The fruit which developed from modified buds also was different from the normal variety and, as with buds, hardly two of a cluster were alike. This variation is illustrated in figure 2 (*B*) and (*C*).

Parthenocarpy. Modified flower buds having the tubular calyx could not open and therefore could not be pollinated. To induce fruit set it was necessary to spray or otherwise treat the buds with β-naphthoxyacetic acid. After treatment the ovary developed rapidly and split and grew out of one side of the tube. As the fruit grew it resembled an acorn [Fig. 2 (*C*)] with a one-piece calyx. The fruit was seedless. β-Naphthoxyacetic acid is not the only growth substance that causes parthenocarpy. Gustafson[5] showed that several of the well-known hormone-like substances induced development of ovaries without pollination. The naphthoxy substances, however, are much more effective in low concentration than the other compounds.

It has been possible to induce seedless fruit of tomato by spraying normal flower clusters with 50 to 300 mg./l. of β-naphthoxyacetic acid. The fruit develops best from flowers which are just open or well-developed buds. If the flower cluster is treated with high concentrations, the parthenocarpically developed fruit may become large and rough with much unfilled space in the locules. Also the style sometimes develops fleshy, giving an unusual point to the fruit. If, however, the lower concentrations, 50 to 300 mg./l., were used, the resulting fruit appeared like the normal for the variety and tasted nearly normal. Most people who were asked to pass judgment on the flavor said they were sweeter than normal.

From early results in the field it appears that if the clusters of flowers are sprayed at the right time the fruit will develop without or with very

few seeds. It seems that if parthenocarpic development occurs before the flower is pollinated, the chances for fertilization of eggs are only slight. This method of inducing fruit set will doubtless be valuable for greenhouse tomatoes where insects are absent and pollination must be effected artificially.

The vapor method may become useful for greenhouse tomatoes. In one experiment potted plants with flowers just beginning to open were placed in various corners of the house. At one end of the house the equivalent of five drops of the ethyl ester of β-naphthoxyacetic acid was vaporized on a warm plate. The heat was stopped after two hours but the greenhouse remained closed all night. The plants showed induced epinasty of leaves within a few hours and within 48 hours development of the ovaries was evident. The flower clusters had four to seven flowers. The largest number of seedless fruit on one cluster was six. The plants in the farthest corner from the hot plate set fruit as well as those nearest the source of vapors.

Bulblets induced on lily stem. Some species of lilies normally produce bulblets on the stem at the axil of the leaf. *Lilium longiflorum* does not possess this characteristic. It was induced, however, to grow such bulblets by spraying the growing tip with an emulsion containing 500 mg./l. of β-naphthoxyacetic acid. The application was repeated three times at seven-day intervals. Roots as well as shoot buds were induced to grow. The adventitious roots came from internodes as well as nodes but the bulblets occurred only in the axil of the leaf (Fig. 3). There were usually two for each node—one on each side not far from the edge. Occasionally one appeared in the middle. In several cases roots appeared in these regions first and prevented bulblets from forming. Beal[4] reported inducing buds in the axils of the upper two or three leaves of decapitated stems with a 3 per cent lanolin preparation of indoleacetic acid.

The fact that the bulblets normally occur in the axil of leaves of many species of lilies indicates that this region normally has a set of cells with special potentialities for buds. Some species need special stimuli in order to make the buds grow. If this assumption is correct, the forced buds of lily are not adventitious in the same sense as bulbils which grow on fleshy scales of lily.

Discussion.—We are in the habit of thinking of the size, shape and pattern of leaves, flowers and fruit as characteristic of a given variety of plants. We think little of what determines these characteristics. The geneticist attributes to the gene the power of controlling the formative nature of plants. Just how the gene functions is not known but the method of regulating growth must be complex. At some stage a chemical influence appears to be involved and environment also plays a part. In nature we find that the pattern of leaves of some water plants varies when grown in

or out of water. Klebs[8] said that living protoplasm had many potentialities and that environment determined which of these can develop. If this view were accepted it would not necessarily militate against the gene theory unless the theory credits the gene with direct power.

The formative responses induced by β-naphthoxyacetic acid can be considered as effects of a modified environment. The degree of response varies

FIGURE 3

Lily to show bulblets and roots induced with emulsion spray containing 500 mg./l. of β-naphthoxyacetic acid. Tip was sprayed three times at seven-day intervals. The plant had two shoots about the same height. When shoot on right was approximately 10 inches high, the tip was treated. Growth was inhibited but roots and bulblets were induced. The enlargements taken about 12 inches above ground.

with the extent to which the environment is changed—that is, the concentration of the chemical. If genes regulate size and form of plants, then the question arises as to whether or not the genes are modified by the chemical. Since the treated plants finally recovered and again produced normal leaves, flowers and fruit, it does not appear likely that the genes are permanently modified. It seems logical to assume, however, that under the influence of the chemical the genes cannot function in a normal way and thus the modified forms occur.

Under the influence of β-naphthoxyacetic acid some of the treated plants exhibited morphological characteristics that suggested certain relationships. For example, the calyxes of fuchsia, which are normally red and pointed, developed green and leaf-like. The involucres of white hydrangea were at first completely inhibited but finally developed green and leaf-like. The modified leaflets of *Mimosa* had projections regularly along one side, suggesting a morphological structure in the normal leaf which is not evident. Tendrils of cucumber which are normally single were modified so that they were branched like some of the related cucurbs. Also, some of the branched tendrils developed leaf-like structures, suggesting that tendrils are modified leaves. The taxonomist and morphologist might be able to find with this method relationships which are otherwise not evident.

The resemblance of modified leaves and flowers to those of plants having a virus disease is very suggestive, since the active agent of the disease is known to be a chemical. The striking difference is that plants recover from treatment with β-naphthoxyacetic acid but not from virus disease. Another similarity is that some species are resistant to both virus and β-naphthoxyacetic acid.

Summary.—Several species of plants were treated with β-naphthoxyacetic acid and the regulatory and formative effects were compared and contrasted with those induced with other growth substances. The principal differences concerned modifications of new organs formed after the plants were treated with β-naphthoxyacetic acid and its derivatives.

The substances were effective when applied in solution or as a vapor. The growing tips were sprayed or the solution was applied to the soil of potted plants. The entire plants were exposed to vapors of the esters of β-naphthoxyacetic acid in bell jars or in the greenhouse.

The new leaves, flowers and fruit which appeared after treatment were modified in size, shape and pattern, and the veins became transparent. The latter response was compared with plants having a virus disease. The treated plants, in time, recovered from the chemical influence and again produced normal leaves and flowers.

The naphthoxy compounds were particularly effective for inducing parthenocarpy and thereby causing seedless fruit to develop. A few drops liberated as vapor at one end of a greenhouse were sufficient to set seedless fruit on flower clusters of tomatoes distributed over the entire house.

[1] Bausor, S. C., "A New Growth Substance, β-Naphthoxyacetic Acid," *Amer. Jour. Bot.*, **26**, 415–418 (1939).

[2] Bausor, S. C., "Response of Tomato Plants to β-Naphthoxyacetic Acid," *Ibid.*, 733–736 (1939).

[3] Bausor, S. C, Reinhart, W. L., and Tice, G. A., "Histological Changes in Tomato Stems Incident to Treatment with β-Naphthoxyacetic Acid," *Ibid.*, **27**, 769–779 (1940).

[4] Beal, J. M., "Histological Responses of Three Species of *Lilium* to Indoleacetic Acid," *Bot. Gaz.*, **99**, 881–911 (1938).

[5] Gustafson, Felix G., "Inducement of Fruit Development by Growth-Promoting Chemicals," *Proc. Nat. Acad. Sci.*, **22**, 628–636 (1936).

[6] Irvine, Virginia Clementine, "Studies in Growth-Promoting Substances as Related to X-Radiation and Photoperiodism," *Univ. Colorado Studies*, **26**, 69–70 (1938).

[7] Killeffer, D. H., "Plant Hormones," *Amer. Chem. Soc., News Edition*, **18**, 395–399 (1940).

[8] Klebs, Georg, "Alterations in the Development and Forms of Plants as a Result of Environment," *Proc. Roy. Soc.* (*Lond.*), **B82**, 547–558 (1910).

[9] Zimmerman, P. W., and Hitchcock, A. E., "Experiments with Vapors and Solutions of Growth Substances," *Contrib. Boyce Thompson Inst.*, **10**, 481–508 (1939).

[10] Zimmerman, P. W., "Formative Effects Induced with β-Naphthoxyacetic Acid," *Ibid.*, **12**, 1–14 (1941).

THE EFFECTS OF WOUNDING AND WOUND HORMONES ON ROOT FORMATION

By Carl D. LaRue

Department of Botany, University of Michigan

Communicated June 30, 1941

Knowledge of the association of roots with wounds is probably ancient. Roots frequently are formed above ringed stems and removal of a ring of bark is common practice in marcottage, or Chinese layering. In ordinary layering a cut is often made on the lower side of the layered stem. Formerly such cuts were supposed to localize food supplies, and they have received a modern interpretation as localizers of growth hormones.

In rooting cuttings Warner and Went[3] found that stems severed by shears root more readily than those cut with a sharp knife. They say, "This might be attributed to a better healing of the wound if a few more cells have been crushed (cf. Haberlandt woundhormones)." Apparently they do not consider that the wounding of the stems has any direct effect on initiation of roots.

No reference has been found to the wound itself as a causal factor in root induction. Observation of root formation in relation to wounds in instances which were not explained by current theories led the author to the assumption that wounding is in itself an initiator of root production, and that wound hormones are the stimulating agents. The following study was made to test this assumption.

Experimental.—The effects of wounding. Leaves of a yellow variety of Coleus were cut off close to the stem. In each experiment the leaves

were made up into two lots matched as to size. Mature leaves proved more satisfactory than young ones because their petioles were longer, and they were less subject to decay during the rooting period. In each experiment the petioles of one set were wounded by repeatedly slicing 2 or 3 cm. of the bases lengthwise with a razor blade until they were reduced almost to a pulp. A control set was left uninjured save for the unavoidable small transverse cut which severed them from the stem. Both sets were kept together in a moist chamber for 48 hours. Then the wounded bases were removed by a transverse cut and an equal amount of petiole was removed from each of the controls. In some tests the leaves were rooted in moist sphagnum, in others on moist paper in glass damp-chambers. No difference could be detected between the results of the two treatments. Some trials were carried out in the weak light of the laboratory; others in full light in the greenhouse. Here again, no differences were noted in results. The data resulting from these experiments are given in table 1.

Trials were made with stems and leaves of several other herbaceous species. Some of these decayed without rooting but others were fairly satisfactory and gave evidence that the effect of wounding is not confined to Coleus alone (table 1).

Woody cuttings did not show increased rooting when wounded; possibly because of the small amount of living tissue which could be wounded.

TABLE 1

EFFECT OF WOUNDING ON ROOT PRODUCTION

SPECIES	PART USED	NUMBER OF TRIALS	NUMBER OF PARTS TESTED	NUMBER OF ROOTS ON WOUNDED CUTTINGS	NUMBER OF ROOTS ON UNWOUNDED CUTTINGS	NUMBER OF WOUNDED CUTTINGS ROOTED	NUMBER OF UNWOUNDED CUTTINGS ROOTED
Coleus Blumei	Leaf	15	580	1788	668	220	161
Mentha arvensis americana	Stem	3	300	1204	593	107	110
Lysimachia nummularia	Stem	2	100	202	72	50	20
Lycopersicon esculentum	Stem	1	18	123	69	9	8
Armoracia rusticana	Leaf	1	30	40	15	4	3
Sedum purpureum	Stem	1	100	521	368	49	48
		23	1128	3878	1785	439	350

The effects of wound hormone extracts. Leaves of plants possessing considerable regenerative power were used for making extracts presumed to contain wound hormones. *Radicula aquatica* has remarkable ability to regenerate (LaRue[2]). *Kalanchoe tubiflorae* and tomato are also remarkably active in root formation. Extracts were made from leaves of these

plants according to the method of Bonner.[1] Coleus leaves treated with these extracts rotted before rooting. On woody cuttings they did not show a toxic effect and numerous tests were made with willow cuttings. The cuttings were allowed to stand in the extract for two days and were then transferred to distilled water. The results are shown in table 2.

TABLE 2

THE EFFECTS OF WOUND HORMONE EXTRACTS ON WOODY CUTTINGS

SPECIES OF CUTTINGS	SOURCE OF EXTRACT	NUMBER OF TRIALS	NUMBER OF CUTTINGS	NUMBER OF ROOTS ON EXPERIMENTAL SET	NUMBER OF ROOTS ON CONTROLS	NUMBER OF EXPERIMENTAL CUTTINGS ROOTED	NUMBER OF CONTROL CUTTINGS ROOTED
Salix fragilis	*Radicula aquatica*	2	140	71	27	41	24
Salix fragilis	*Lycopersicon esculentum*	4	280	143	80	84	61
Salix fragilis	*Kalanchoe tubiflorae*	2	200	287	127	94	59
Populus Bolleana	*Radicula aquatica*	1	100	29	6	22	6
Lycopersicon esculentum	*Kalanchoe tubiflorae*	1	86	140	75	10	7
Muehlenbeckia platyclada	*Kalanchoe tubiflorae*	1	100	36	5	11	2
		11	903	606	320	262	159

The other species included in table 2 were less satisfactory than willow but also showed a stimulus due to treatment with the wound hormone extracts.

Wounding and treatment with wound hormones both increase the total number of roots produced. Wounding has usually more than doubled the number of roots, and treatment with wound hormones has had nearly as great an effect. Both methods of treatment have increased the number of cuttings rooted but this effect is not nearly so great as that on total number of roots formed.

Effect on rate of root formation. It was found that root formation took place earlier on wounded cuttings and on cuttings treated with wound hormones than on controls. Later the differences in numbers of roots on treated and control cuttings were greatly reduced by root formation on controls. This is demonstrated by tables 3 and 4.

Observation indicated that the roots on the experimental cuttings were larger and longer than on controls but further study is needed to make certain of this.

Comparison of the effects of wounding and of indole butyric acid. In this trial leaves of the yellow variety of Coleus were used. In each experiment four sets of matched leaves were used. One set was untreated for control;

TABLE 3

EFFECT OF WOUNDING ON RATE OF ROOT DEVELOPMENT ON PETIOLES OF YELLOW COLEUS

EXPERIMENT	NUMBER OF DAYS AFTER PLANTING	WOUNDED PETIOLES	CONTROLS
1	5	20	2
	9	76	18
2	7	39	1
	9	140	82
3	7	47	0
	10	157	65
4	7	75	3
	9	91	29
Totals	5–7	181	6
	9–10	464	194

TABLE 4

EFFECT OF WOUND HORMONE EXTRACTS ON RATE OF ROOT DEVELOPMENT ON CUTTINGS OF SALIX FRAGILIS

EXPERIMENT	NUMBER OF DAYS AFTER PLANTING	EXPERIMENTAL SET	CONTROLS
1	11	129	23
	18	143	80
2	8	14	4
	28	231	179
3	8	23	9
	12	47	31
Totals	8–11	166	36
	12–28	540	290

TABLE 5

COMPARATIVE RESULTS OF WOUNDING AND THE APPLICATION OF INDOLE BUTYRIC ACID

TRIAL	NUMBER OF LEAVES	TREATMENT	NUMBER OF ROOTS
1	25	Controls	26
	25	Wounded at base of petiole	296
	25	Indole butyric acid at tips of leaves	396
	25	Wounded and indole butyric acid at tips of leaves	232
2	25	Controls	43
	25	Wounded at base of petiole	85
	25	Indole butyric acid at tips of leaves	70
	25	Wounded and indole butyric acid at tips	149
3	25	Controls	58
	25	Wounded at base of petioles	171
	25	Indole butyric acid at tips of leaves	229
	25	Wounded and indole butyric acid at tips of leaves	202

one had the bases of petioles sliced and bruised; one had 1% indole butyric acid in lanolin applied to the upper surface of the tips of the leaves; and a fourth had the bases of the petioles wounded and 1% indole butyric acid applied to the leaf tips. The results are given in table 5.

The effect of wounding appears to be nearly as great as that of the application of indole butyric acid. The combined effect of wounding and the use of indole butyric acid is not greater than that of indole butyric acid alone.

Summary.—1. In petioles of Coleus and in herbaceous cuttings of some other species wounding greatly increases the production of adventitious roots.

2. Treatment with extracts which contain wound hormones strongly stimulates root formation in woody cuttings.

3. Both wounding and treatment with extracts containing wound hormones shorten the time required for rooting of cuttings.

4. The effect of wounding on root initiation appears to be nearly as great as that of treatment with indole butyric acid.

[1] Bonner, J., *Proc. Nat. Acad. Sci.*, **22**, 426 (1936).

[2] LaRue, C. D., Papers, *Mich. Acad. Sci., Arts and Let.*, **22**, 123–139 (1936).

[3] Warner, C. C., and Went, F. W., *Rooting of Cuttings with Indole Acetic Acid and Vitamin B_1*, pp. 1–12, Plant Culture Publishing Co., Pasadena, California (1939).

STERILITY IN CROSSES OF GEOGRAPHICAL RACES OF DROSOPHILA MICROMELANICA[1]

BY A. H. STURTEVANT AND E. NOVITSKI

WILLIAM G. KERCKHOFF LABORATORIES OF THE BIOLOGICAL SCIENCES, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated June 16, 1941

Drosophila micromelanica Patterson is a member of the subgenus Drosophila, related both to *D. melanica* and to *D. mulleri*. It has been found by Professor J. T. Patterson "throughout central Texas," and two separate collections have been made in the Huachuca Mountains of southern Arizona (C. Epling, G. Mainland).

We have studied two Texas strains (Austin, Smithville—both from Professor Patterson) and two from Arizona (collected by Professor C. Epling). These are phenotypically indistinguishable and have identical metaphase chromosome groups (five pairs of rods that are approximately equal in length, and a pair of dots).

Crosses have shown that Arizona ♀ × Texas ♂ yields fertile males and females, but Texas ♀ × Arizona ♂ yields fertile females and sterile males.

Further experiments indicate that the sterility of these males is dependent on their possession of a Texas *X* and an Arizona *Y* (cf. the similar case in *D. macrospina*, briefly reported by Mainland[2]).

In the Austin stock two sex-linked mutant types have been found—white and echinus. Linkage tests have given 49.1% (285/580) recombination between the genes concerned. An Arizona wild-type chromosome, tested against a (Texas) white echinus one, gave 58/123 = 47.9% recombination. The fathers in this case were Arizona males, the mating being thus Arizona/*w ec* × Arizona. Tests were made on some of the sons from this mating, with the result shown:

	+	*ec*	*w*	*w ec*
Fertile	7	9	1	0
Sterile	1	0	8	7

Evidently the difference between the two *X*'s lies near the locus of white—this being the critical evidence that *X* is concerned. The evidence that *Y* is concerned is less direct, being derived chiefly from the observation that, in a number of different complex experiments, strains that were largely Arizona in origin but had Texas *Y*'s have given no sterile males when crossed to Texas females.

The nature of the sterility in these males is not clear. They regularly inseminate females, and such females are found to have active sperm in their receptacles. The resulting eggs have never been observed to hatch, and no considerable development occurs in them. It remains uncertain whether they are fertilized, and whether any cleavage divisions occur.

Preliminary observations on the salivary gland chromosomes of Texas-Arizona hybrids indicate normal pairing of homologues, and inversions in only two of the four long autosomes that are present. In one of these there is a single inversion; in the other there are three, of which two are overlapping and the third is independent of both of these. There are no inversions in the *X*. The genetic data also indicate that there are no inversions in the *X*, since the recombination frequencies recorded above are not significantly different in pure Texas females and in hybrids. Hybrid females have also given a total of 235 sons in matings where patroclinous males would have been detected, and no such males have been seen. It should be added that no systematic survey has been made of the salivary gland chromosomes; other hybrids may show more or fewer inversions than the ones studied.

Summary.—Strains from central Texas and from southern Arizona are not distinguishable, but when they are crossed the hybrid males from Texas ♀ × Arizona ♂ are sterile; all other F_1 hybrids are fully fertile. The sterility is due to the presence of a particular portion of a Texas *X* chromosome and (probably) an Arizona *Y*. The sterile males inseminate

their mates with active sperm, but no larvae develop in eggs laid by these females.

[1] This species is undescribed. Professor J. T. Patterson has furnished the following brief description. A more complete one will be published later.

"*Drosophila micromelanica*, sp. nov.

"A small, very dark melanica-like species with the following general characters: Arista with eight branches; acrostichal hairs somewhat irregular, in six to eight rows; no prescutellars. Spermathecae almost transparent, not chitinized; ventral receptacle emerges from uterus as straight tube and then forms a tangled mass, without definite coiling. The eggs have two filaments. Length of body 2.4 mm.; wing 3 mm. (measurements made on live specimens)."

[2] Mainland, G. B., *Genetics*, **26**, 161 (1941).

LINKAGE STUDIES OF THE RAT (RATTUS NORVEGICUS) V

BY W. E. CASTLE AND HELEN DEAN KING

UNIVERSITY OF CALIFORNIA AND WISTAR INSTITUTE

Communicated July 7, 1941

I. The Albino Chromosome.—The rat chromosome which contains the largest number of known mutant genes is the one which carries the albino gene. Albinism was probably the first observed mutation of the rat and albinos had long been reared in captivity before the Mendelian laws were known. Albinism in the rat, as in other rodents, is a simple recessive in heredity, as are also the four other mutations which are linked with it. The five mutant genes (as we shall presently show) are in the order of their location in the chromosome (1) *l*, a lethal recently discovered by Grüneberg; (2) *c*, albino; (3) *r*, red-eyed yellow; (4) *p*, pink-eyed yellow; and (5) *w*, waltzing. The heredity of *c*, *r* and *p* was studied by Castle, Wright, Dunn and Wachter, the first observed case of linkage in the rat having been reported by Castle and Wright as occurring between *r* and *p*. Further intensive studies made by Castle, Dunn and Wachter, showed that *r* was closely linked with *c* with less than one per cent of crossing-over between them and that *p* was more loosely linked with *c*. The fact was also established that crossing-over in this chromosome (as in the albino chromosome of mice) occurs with greater frequency in female than in male individuals. The linkage maps involving the three genes were accordingly drawn thus:

For females	*C*	*R*	*P*
	0	0.5	21.9
For males	*C*	*R*	*P*
	0	0.2	18.4

After King had discovered the waltzing mutation in a race of albino rats, it was found that a loose linkage exists between waltzing and albinism, with about 45 per cent of crossing-over. This would locate the waltzing gene as far to the right of *P* in the above maps, as *P* is to the right of *C*, unless *w* lies to the left of *c*, in which case the total map length of the chromosome would be considerably greater. To decide between these two alternatives the present investigation was made.

Grüneberg on the basis of studies of linkage between *l*, *c* and *p*, concluded that his lethal lies 3.3 units to the left of *c*, and estimated the total length of the albino chromosome at 49 units.

To ascertain whether *w* lies to the right or to the left of *c* it is necessary to have evidence of the linkage strength between *w* and some third gene of the albino chromosome. For this purpose gene *p* is most suitable since it lies at a considerable distance from *c*. If the linkage between *p* and *w* is *less* than that between *c* and *w*, then *w* must lie to the right of *c;* but if the linkage between *p* and *w* is *greater* than that between *c* and *w*, then *w* must lie to the left of *c*. To answer this question King has made a repulsion cross between a normal pink-eyed yellow and a waltzing albino. The resulting F_1 animals were gray non-waltzers of constitution $\frac{C\ p\ W}{c\ P\ w}$. They were mated to triple recessive animals, albino waltzers born of pink-eyed yellow parents, and so of constitution *cc pp ww*, which in every gamete would transmit the three recessive alleles, *c p w*.

The back-cross population produced in this way consisted of 523 individuals as follows:

	NORMAL	WALTZING	TOTAL
Gray	49	11	60
Pink-eyed yellow	180	42	222
Albino	175	66	241
Total	404	119	523

From these data we desire to estimate the crossover percentage (1) between *C* and *P*, (2) between *P* and *W* and (3) between *C* and *W*. Let us assume first that the order of the genes is *C P W*. The constitution of the F_1 animals has already been stated. A crossover between loci *C* and *P*, will produce gamets *C P* (gray) and *c p* (albino). But since these albinos are not distinguishable from other albinos produced in the experiment, it will be best to disregard them and base the calculation of the crossover percentage solely on the gray class which is distinctive and theoretically should equal half the total number of crossovers in this region. Accordingly $2 \times 60 = 120$ will be the number of crossovers in the population of 523, and this equals 22.9 ± 1.2 per cent. This agrees fairly well with the result obtained by Castle and Wachter in a much larger population which was 21.9 ± 0.4 (for female F_1 parents).

A crossover between loci *P* and *W* in the F_1 parent will result in gametes which are *C p w* (yellow waltzer) or *c P W* (albino normal). Disregarding the albinos, the total crossovers should equal 2 × 42 (yellow waltzers) = 84 = 16.0 per cent. There is reason to think this estimate too low, because the waltzers recorded in the entire population fall considerably below the expected fifty per cent. A like deficiency of waltzers was observed in King's earlier experiments with this character. It is undoubtedly due to the low effectiveness ("penetrance") of the waltzer gene, which results in "normal overlaps," animals which though homozygous for the waltzer gene nevertheless do not waltz. The expected number of waltzers (half the total population) is 2.2 times as great as the observed number. Accordingly to correct for the deficiency of waltzers, we should multiply by 2.2 the observed numbers of waltzers, and decrease correspondingly the reported number of normals. If this is done the population statistics become

	NORMALS	WALTZERS	TOTAL
Gray	35.8	24.2	60
Yellow	129.6	92.4	222
Albino	95.8	145.2	241
Total	261.2	261.8	523

From the data thus corrected, the crossovers between loci *P* and *W* would be 2 × 92.4 = 184.8 = 35.3 per cent of the population 523.

The crossover percentage between *C* and *W* may also be estimated from the corrected data. Crossover individuals would be colored waltzers and albino normals, but since the former are more distinctive, the calculation becomes 2 × 116.6 (colored waltzers) = 233.2 = 42.6 per cent. This is in fair agreement with the estimate of King and Castle based on a much larger population, which was 45.8 ± 0.7 per cent. That the crossover percentage is less in this experiment than in the earlier one conducted by King may be explained by the known greater frequency of crossing-over in female than in male F_1 parents. In the present experiment all but 49 of the 523 young were sired by F_1 males, whereas in the earlier experiment a larger proportion of the population was derived from F_1 females.

We are now able to ascertain the order of the three genes *C*, *P* and *W*. The indicated map distance *P W* (35.3) is less than *C W* (42.6). Accordingly *P* lies between *C* and *W* and the respective distances between the genes may be indicated thus:

$$\frac{C\ 22.9\ P\ 35.3\ W}{42.6}$$

C P + *P W* = 22.9 + 35.3 = 58.2 which is greater than *C W* (42.6) because *C W* is made to appear less by double crossing-over (interference).

If instead of the figures for the *C P* and *C W* distances given by this

investigation, we adopt those given by earlier investigations based on larger numbers, and also add to the map the portion discovered by Grüneberg, to the left of *C*, then the map will become

L	*C*	*R*	*P*	*W*
0	3.3	3.8	23.3	58.6

II. The Curly-Brown Chromosome.—A second chromosome of the rat in which more than one mutant gene had been found was reported by King and Castle (1935). Here a loose linkage was demonstrated to occur between the dominant gene curly (*Cu*) and the recessive gene brown (*b*), the crossover percentage being estimated at 40.48 ± 1.35.

A third gene anemia (*an*) discovered by Smith and Bogart (1939) has been found to be located in this same second (II) chromosome. Since anemia is a lethal, only animals heterozygous for anemia can be used in linkage studies. Castle has made a cross between such carriers of anemia and curly individuals. F_1 curly individuals, which were found by a breeding test to be carriers of anemia, would carry the two genes in the repulsion relationship. In a backcross of such F_1 animals to non-curly carriers of anemia it was found that very few of the anemic individuals thus produced were also curly coated, an indication of close linkage (repulsion) between the genes for anemia and curly. By suitable crosses (though not without difficulty because of the close linkage between curly and anemia) several triple heterozygotes were finally produced which transmitted all three mutant genes in the same chromosome, their genetic formula being $\frac{Cu\ an\ b}{+\ +\ B}$. Such animals were now mated with animals homozygous for brown but heterozygous for anemia, this being the nearest approach to a triple recessive combination which would be viable. Such backcross matings produced populations as follows:

	Anemic Phenotypes				
	Cu an B	*Cu an b*	*an B*	*an b*	TOTAL
By F_1 males	129	187	1	6	323
By F_1 females	25	32	0	1	58
Total	154	219	1	7	381
	Normal Phenotypes				
	Cu B	*Cu b*	*B*	*b*	TOTAL
By F_1 males	124	159	257	276	816
By F_1 females	28	47	49	49	173
Total	152	206	306	325	989

Considering first the anemics, it will be observed that only the 7 *an b* individuals and the one *an B* individual have resulted from a crossover in the F_1 parent between *Cu* and *an*. They comprise 2.1 ± 0.5 per cent of the anemics.

Crossovers between anemia and brown are represented by the black anemics (whether or not curly), which number 155, or 40.7 ± 1.7 per cent of the anemics.

Crossovers between curly and brown are represented by curly blacks and non-curly browns, which among the anemics number 154 + 7 = 161, or 42.2 ± 1.7 per cent of the anemics. A diagram expressing these relationships may be drawn thus

$$\frac{Cu\ 2.1\ an\ 40.7\ b}{42.2}$$

Obviously the order of the genes is, as previously assumed, *Cu an b*.

The non-curly phenotypes give additional evidence of the frequency of crossing-over between the genes *Cu* and *b*, since here also curly blacks and non-curly browns will arise from crossover gametes produced by the F_1 parent. They number 152 + 325 = 477 in a population of 989, or 48.2 ± 1.07 per cent of the population, a somewhat higher value than the anemics alone gave. Combining the evidence from both sets of data, we have 638 crossovers in a population of 1370, which is 46.56 ± 0.92 per cent.

Additional evidence on the crossover percentage between curly and brown is furnished by backcross litters which contained no anemics because the backcross mate of the F_1 animal did not carry anemia. Such matings yielded only normal phenotypes, as follows:

Cu B	*Cu b*	*B*	*b*
86	97	96	91

The crossovers = 86 + 91 = 177, which is 47.8 ± 1.7 per cent of the population, 370.

The data recorded by King, on which the original discovery of linkage between curly and brown was based, included 253 crossovers in a population of 625, which gives a crossover percentage of 40.48 ± 1.30.

If now we combine all available evidence we obtain a record of 1068 crossovers in aggregate population of 2365, or 45.16 ± 0.69 per cent crossovers.

According to our present knowledge the second (II) chromosome of the rat may be mapped thus:

$$\frac{Cu\ an\ b}{0\quad 2\quad 45}$$

Castle, W. E., and Sewall Wright, *Science*, **42**, 193–195 (1915).
Castle, W. E., Publ. No. 241 Carnegie Inst. Wash., 175–180 (1916).
————, Publ. No. 288 Carnegie Inst. Wash., 29–55 (1919).
Dunn, L. C., *Genetics*, **5**, 325–343 (1920).
Castle, W. E., and W. L. Wachter, *Ibid.*, **9**, 1–12 (1924).
King, Helen Dean, and W. E. Castle, *Proc. Nat. Acad. Sci.*, **21**, 390–399 (1935).
————, *Ibid.*, **23**, 56–60 (1937).
Smith, S. E., and R. Bogart, *Genetics*, **24**, 447–493 (1939)
Grüneberg, H., *Ibid.*, **23**, 732–746 (1939).

MAXIMAL SUBGROUPS WHOSE ORDERS ARE DIVISIBLE BY TWO OR THREE

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

Communicated July 7, 1941

A necessary and sufficient condition that a given invariant subgroup is a maximal subgroup is that it is of prime index under the group G. In this case the difficulties involved relate mainly to the properties of this subgroup. Hence we shall restrict our attention in the present article to maximal subgroups which are non-invariant under G. These non-invariant maximal subgroups may appear in various sets of conjugate subgroups. It will be convenient to restrict our attention here to one such set of n subgroups. These subgroups are transformed under G according to a transitive permutation group which is primitive because the subgroups are maximal. When every pair of these subgroups has only the identity in common this primitive group is of degree n and of class $n-1$ and hence it belongs to a category of groups which has received much attention. The fact that one of these maximal subgroups cannot transform another into itself results from the restriction to non-invariant maximal subgroups in the present article.

If two of these maximal subgroups had a common operator besides the identity G would not be of class $n-1$ since this common operator would transform each of these two subgroups into itself. According to a well-known theorem due to G. Frobenius (1849–1917) G therefore contains a regular invariant subgroup generated by its permutations of degree n and the corresponding quotient group is simply isomorphic with one of the given n conjugate maximal subgroups. All the operators of G which correspond to the same operator in this quotient group are conjugate under G and hence they are of the same order. When this order is 2, each of these operators transforms every operator of the given regular invariant subgroup into its inverse and hence this subgroup is abelian according to the theorem that if every operator of a group corresponds to its inverse in an automorphism of the group then this group is abelian.

Suppose that one of the operators of the quotient group of G with respect to the given regular invariant subgroup is of order three. All the operators of G in the co-set which corresponds to this operator of order 3 are then also of order 3 and constitute a single set of conjugates under G. If s_1 and s_2 are two such operators then $s_1s_2s_3$ is in the given invariant regular subgroup of G and is transformed into operators which are commutative with it by s_1 and s_2, as can readily be verified. As $s_1s_2s_3$ is an arbitrary operator in the given invariant subgroup and is transformed into $s_2s_1^2$ by s_1 as well as by s_2 it

results that the abelian group generated by the two operators $s_1s_2s_3$ and $s_2s_1^2$ is invariant under G. That is, *if a group contains a complete set of conjugate non-invariant maximal subgroups every two of which have only the identity in common and are of an order which is divisible by either two or three then the group involves an abelian invariant subgroup whose order is prime to the order of these subgroups.*

The primitive permutation groups of odd degree furnish many applications of this theorem whenever their subgroups composed of all their permutations which omit one letter are regular. For instance, the doubly transitive group of degree 25 in which the group composed of all the permutations of degree 24 together with the identity constitute the cyclic group of order 24 has an order which is divisible by both two and three and hence the given regular invariant subgroup is the non-cyclic group of order 25. Similarly, in the doubly transitive group of degree 9 and of order 72 in which the subgroup formed by the permutations which omit one letter is the cyclic group of order 8 has for the invariant regular group of order 9 the non-cyclic group of this order. In this case the order of the maximal subgroup in question is divisible only by the prime number 2 while in the preceding case the order of the corresponding maximal subgroup is divisible by both 2 and 3.

When two abelian maximal subgroups of a given group have two operators in common these operators constitute a subgroup of the central of the group. The corresponding quotient group will then have a complete set of conjugate maximal subgroups such that every two of the subgroups of the set have only the identity in common and hence these subgroups of the quotient group come under the theorem noted above. When two maximal subgroups which are not abelian have common operators and these operators constitute an invariant subgroup of the group then the corresponding quotient group will again have a set of conjugate maximal subgroups such that every two have only the identity in common and hence will also come under the theorem noted above. If the subgroup composed of all the operators which are common to two maximal subgroups of a group is invariant under one of these maximal subgroups without being invariant under the other then the former must contain at least two other subgroups which are simply isomorphic with this invariant subgroup because they are invariant under conjugates of the former subgroup.

In a multiply transitive permutation group of degree n it is not possible that two of the n maximal subgroups of degree $n - 1$ which are separately composed of all the permutations of the group which omit a given letter have a cross-cut which is invariant under one of these two maximal subgroups but not under the other since an invariant subgroup of a transitive permutation group involves all the letters of the group unless it is the identity. On the contrary, in a simply transitive primitive permutation

group two such maximal subgroups may have a cross-cut which is invariant under one of them without being also invariant under the other. In fact, this is the case when the symmetric group of degree 5 is represented as a simply transitive primitive group of degree 10 with respect to its intransitive subgroup of order 12. This subgroup is clearly a maximal subgroup of the symmetric group of degree 5.

In this primitive group of degree 10 and of order 120 the maximal subgroup of order 12 and of degree 9 has for its transitive constituents a group of degree 6 and of order 12 and the symmetric group of degree 3. It therefore contains an invariant subgroup of order 2 and of degree 6 which appears in three of its other conjugates but is not invariant under one of these conjugates. Hence there results the following theorem: *In a multiply transitive permutation group of degree n it is not possible that the cross-cut of two of its maximal subgroups which are composed separately of all its permutations which omit one of its letters is invariant under one of these two conjugate subgroups but not under the other, while in a simply transitive primitive permutation group such maximal subgroups may appear.* In view of the fact that comparatively little is known in regard to the simply transitive primitive permutation groups this theorem may deserve some emphasis. It should be noted that the maximal subgroups composed of all the permutations which separately omit one letter of the simply transitive primitive group composed of the positive permutations of given group of order 120 do not have the property that their cross-cut is invariant under one but not under the other of two conjugates.

By definition a maximal invariant subgroup is an invariant subgroup which corresponds to a simple quotient group. If this simple group is of prime order the corresponding maximal invariant subgroup is also a maximal subgroup but if it is of composite order the corresponding maximal invariant subgroup is not a maximal subgroup since a maximal subgroup of a group is necessarily of prime index under the group. If a group has two maximal invariant subgroups which are also maximal subgroups their cross-cut is the direct product of the corresponding two quotient groups. It was in connection with such subgroups that the concept maximal invariant subgroup was introduced by C. Jordan (1838–1922) since the group of a solvable equation necessarily contains a maximal invariant subgroup which is also a maximal subgroup. More recently maximal subgroups received additional attention in view of the fact that the cross-cut of all the maximal subgroups of a group is its ϕ-subgroup.[1]

In a transitive permutation group of degree n in which the subgroup composed of all the permutations which omit one letter is of degree $n-1$ the cross-cut of all the maximal subgroups is clearly the identity. When this subgroup is of degree $n-\alpha$, $\alpha > 1$, there are exactly α permutations on the n letters of this transitive group which are commutative with every

permutation of this transitive group. All of these permutations besides the identity are of degree n and the given cross-cut has an order which is divisible by α since this cross-cut is an invariant subgroup of the given transitive permutation group. A necessary and sufficient condition that the subgroup composed of all the permutations of this group which omit a given letter is maximal is that it cannot be extended by a permutation of the group so that the extended group is intransitive.

[1] Miller, Blichfeldt, Dickon, *Finite Groups*, John Wiley & Sons, p. 71 (1916).

METRIC FOUNDATIONS OF GEOMETRY

By Garrett Birkhoff

Department of Mathematics, Harvard University

Communicated July 9, 1941

1. Introduction.—In the following, a set of metric postulates will be developed for n-dimensional Euclidean, hyperbolic and spherical geometry in the large. It is well known that any such space S satisfies the following postulates:

I. S is a metric space.

II. Any two points of S can be joined by a straight segment; and any two sufficiently short straight segments issuing from the same point a of the same length, have either no other points or all other points in common.

III. Any isometry between portions of S can be extended to a self-isometry of all of S.

What is not realized is that, conversely, any space satisfying these postulates is isometric with one of the models described above. A similar result has been proved by Busemann,[1] but under additional hypotheses including finite compactness which excluded the spherical case.

In the author's opinion, however, the interest of the results below is not so much in the actual result mentioned as in the elementary character of the proofs of the theorems stated. In particular, the arguments in the bounded case (n-dimensional spherical geometry) are of a new type.

2. Preliminary Definitions and Results.—It is well known that, in any metric space, one can define such things as spheres, straight segments, geodesics (curves locally straight), bounded sets, diameters, cluster points, homeomorphisms, isometries and so on. The definitions of these terms will not be repeated here. We also note that Postulates I–II hold in any

Riemannian geometry; thus it follows that it is Postulate III of Pasch which distinguishes Euclidean, hyperbolic and spherical geometry from other Riemannian geometries.

THEOREM 1. *Through any two points passes an infinite geodesic. This is a straight line if space has infinite diameter; if space has finite diameter d, its segments of length d are straight. Any geodesic is homogeneous, and any two geodesics are isometric.*

DEFINITION. If the distance between two points is the diameter of space, they will be called antipodal.

THEOREM 2. *If two points can be joined by more than one straight segment, they are antipodal, and space is bounded. Conversely, if space is bounded, then every geodesic is periodic, of period twice the diameter of space.*

THEOREM 3 (DIGRESSION). *If the second part of Postulate II is replaced by the hypothesis that there exist two points joined by only one straight segment, then either Postulate II holds or space is topologically one-dimensional and not locally compact.*

3. Angles and Perpendicularity.—By an "angle" is meant the configuration formed by a point and two half-lines issuing from that point; straight angles and equal (isometric) angles are also easy to define.

THEOREM 4. *If two straight lines intersect, the opposite angles are equal.*

THEOREM 5. *The correspondence which reflects each geodesic through a point a in a is an isometry.*

It is called reflection in the point a.

DEFINITION. An angle is a "right angle" if the cross formed by extending the half-lines beyond the vertex admits eight isometries.

THEOREM 6. *The segment from a point p to the midpoint of a segment of a line L makes a right angle with L if and only if p is equidistant from the endpoints of the segment.*

THEOREM 7. *In the unbounded case, one and only one perpendicular can be dropped from a point p to a geodesic L. In the bounded case, either every line through p is perpendicular to L, or the perpendiculars are equally spaced.*

THEOREM 8. *Two triangles are isometric if their sides are equal in pairs (unless one side is of length d), or if two sides and the included angle are equal.*

THEOREM 9. *All right angles are equal.*

4. Flats and Subspaces.—A "subspace" of S is defined as a subset which also satisfies Postulates I–III relative to the metric in S; a "flat," as a subset which contains, with any two non-antipodal points, the unique geodesic through them.

Clearly any geodesic is a subspace, any subspace is a flat and any intersection of flats is a flat. It is also almost trivial that the set of all points left invariant by any isometry is a topologically closed flat.

DEFINITION. A "hyperplane" is the set of all points equidistant from two suitable fixed points.

THEOREM 10. *Any hyperplane divides space in two, and is a maximal flat. Any two hyperplanes are isometric.*

THEOREM 11. *There is just one isometry besides the identity which leaves a given hyperplane pointwise invariant.*

This isometry is called *reflection* in the hyperplane. In somewhat the same vein, one can prove that the group of isometries leaving a given hyperplane setwise invariant is transitive on the points of the hyperplane, and also on those at any given positive distance from the hyperplane.

THEOREM 12. *In the case S has finite diameter d, there are just two points whose distance from a given hyperplane is $\frac{1}{2}d$; these are antipodal.*

THEOREM 13. *In the bounded case each point admits one and only one antipodal point.*

This basic fact yields many corollaries.

THEOREM 14. *Any hyperplane is a subspace, or consists of two antipodal points.*

The facts just proved suggest modifying our definitions of subspace and flat to simplify the statement of theorems, as follows. In the bounded case, the set consisting of two antipodal points will be admitted as a degenerate subspace, and a flat will be required to contain with each point, its antipode (as it does unless it consists of just one point).

5. Orthocrosses and Dimension.—By an "orthocross" we mean a set of mutually perpendicular lines (called "axes") through a common vertex. A maximal orthocross will be called a "rectangular basis."

By Theorem 9, any one-one correspondence between the axes of two orthocrosses can be induced by an isometry; also, any space has a rectangular basis. The following is less trivial.

THEOREM 15. *Any two rectangular bases of S contain the same finite number of axes.*

This integer is called the *dimension* of S. It is not hard to prove that the dimension of any hyperplane of S is one less than the dimension of S; this fact enables one to make arguments by induction on dimension.

DEFINITION. By $p \perp q$ from o, we mean that reflection in o carries p into a point at the same distance from q.

With one exception, this means that the angle made by the lines $\overline{op}$ and $\overline{oq}$ is a right angle.

Now let o be fixed; for any set R, we let R^* denote the set of all p such that $p \perp r$ from o for all $r \in R$. General arguments show[2] that the correspondence $R \rightarrow (R^*)^*$ is a closure operation, while if R is "orthoclosed" in the sense that $R = (R^*)^*$, then R and its "polar" R^* are complements in the lattice of all orthoclosed sets.

DEFINITION. Two orthocrosses are "complementary" if they have the same vertex and their sum is a rectangular basis.

LEMMA. If R and R^* are polar orthoclosed sets, then R and R^* contain complementary orthocrosses with vertex o.

THEOREM 16. *Let θ and θ' be complementary orthocrosses. Then the intersection of all flats containing θ is the polar $(\theta')^*$ of θ', the orthoclosure $(\theta^*)^*$ of θ and a subspace.*

THEOREM 17. *Let R and R^* be polar from o. There exists one and only one isometry ϕ which leaves R pointwise invariant and reflects R^* in o. It is involutory.*

The isometry ϕ is called *reflection in R.*

THEOREM 18. *The only isometries of period two are reflections in subspaces and the "antipodal" involution, which carries each point into its antipode.*

THEOREM 19. *Apart from trivial exceptions, the following conditions on a set T are equivalent: T is a flat, T is a subspace, T is orthoclosed, T consists of the fixpoints of a suitable isometry.*

6. Lattice-Theoretic Structure.—We now derive the known[3] lattice-theoretic structure of a space from our postulates, without reference to coordinate representation.

THEOREM 20. *The lattice of subspaces of any unbounded n-dimensional space is a matroid lattice of dimension $n + 1$.*

Using the polarity operation of complementation, and the known fact that any self-dual matroid lattice is modular, we get the following sharper result.

THEOREM 21. *The sublattice of those subspaces which contain any fixed point o is an abstract projective geometry.*

In the bounded case, one can sharpen the results by an ingenious special definition. Define $p \rho q$ to mean that the distance from p to q is half the diameter of space. Then construct polars T^* with respect to *this* relation. It is easy to show that for any set T, T^* is an intersection of hyperplanes, and hence a subspace—and, conversely, that if T is a subspace, then $T = (T^*)^*$. The complementation operation is no longer restricted to subspaces through o.

THEOREM 22. *If space is bounded, the subspaces form an abstract projective geometry.*

7. Further Results.—The author plans to develop the methods used in proving the above theorems far more extensively, and to publish detailed proofs of all results elsewhere. But some further remarks should be added to the present note.

In the first place, elementary inductive arguments based on the preceding results prove that in the unbounded case, S is homeomorphic with Euclidean n-space, and in the bounded case, with the n-sphere—whence in any case, any closed bounded portion of S is compact. Moreover the appropriate properties of parallelism follow even more immediately.

What is more important, definitions of the sum and of the relative magnitude of two angles can be set up, by restricting one's attention to two-dimensional subspaces. One can then prove that angles are *quantities*, which can be measured by real numbers modulo 360. Following this, one can follow standard methods in separating the three cases of Euclidean, hyperbolic and spherical geometry according to the sum of the three angles of a triangle. By standard methods,[4] one can then set up the trigonometric functions and establish the isometry in the large with the standard geometries.

[1] Busemann, H., "On Leibniz's Definition of Planes," *Am. Jour.*, **63**, 101–111 (1941).

[2] Cf. the author's *Lattice Theory*, New York, 1940, §32.

[3] Menger, K., "New Foundations of Projective and Affine Geometry," *Annals of Math.*, **37**, 456–482 (1936).

[4] For the standard methods referred to, cf. Coolidge, J. L., *The Elements of Non-Euclidean Geometry*, Oxford, 1909, Chaps. III–IV.

LINEAL ELEMENT TRANSFORMATIONS WHICH PRESERVE THE ISOTHERMAL CHARACTER

By Edward Kasner

Department of Mathematics, Columbia University, New York, N. Y.

Communicated July 8, 1941

1. Introduction.—It is a famous theorem in the theory of functions of a complex variable that the conformal transformations are the only *point* correspondences of the plane which carry all isothermal families into isothermal families of curves. Although this result is well known, it appears that there are no known generalizations in the plane. In the present paper, it is our intent to generalize this result to *lineal element* transformations. We also present a discussion of a new system of curves associated with any arbitrary field: the additive-multiplicative trajectories. The isogonal trajectories lead to the very interesting Cesaro-Scheffer theorem, whereas these new trajectories possess the Property I which first presented itself in my study of dynamical trajectories. See our two italicized theorems below.

2. Isothermal Families into Isothermal Families.—A lineal element consists of a point and a direction through that point. Any such element may be most conveniently denoted by the number triplet (x, y, θ), where (x, y) are the cartesian coördinates of the point and θ is the inclination of the direction. We shall call the totality of ∞^3 lineal elements of the plane the *opulence*.

A field is said to be isothermal if its ∞^1 integral curves are conformally equivalent to a parallel pencil of straight lines. The field $\theta = \theta(x, y)$ is isothermal if and only if θ is a harmonic function, that is, θ satisfies the Laplace equation $\theta_{xx} + \theta_{yy} = 0$. The preceding statements are equivalent to the usual definition of an isothermal system. The set of ∞^1 curves $F(x, y) = \text{const.}$ is isothermal if F is a harmonic function.

Any lineal element transformation carries any field into a field, but it does *not* preserve the isothermal character. After some considerable work, the following proposition may be proved.

FUNDAMENTAL THEOREM. *The complete group of lineal element transformations which convert all isothermal families into isothermal families is*

$$X = \phi(x, y),\ Y = \psi(x, y),\ \Theta = a\theta + h(x, y), \tag{1}$$

where ϕ and ψ satisfy the Cauchy-Riemann equations (direct or reverse)

$$\phi_x = \pm\psi_y,\ \phi_y = \pm\psi_x, \tag{2}$$

and h is any harmonic function of (x, y), and a is a non-zero constant.

The contact part of this group is the conformal group. That is, the only contact transformations which preserve isothermal character constitute merely the conformal group.

Thus any transformation of this group may be factored into a conformal transformation, followed by the lineal element transformation

$$X = x,\ Y = y,\ \Theta = a\theta + h(x, y), \tag{3}$$

where a is a non-zero constant and h is a harmonic function of (x, y).

3. Turns and Magniclines.—In the preceding section, we have stated that the group of lineal element transformations which preserves the isothermal character is given by (1). The subgroup which leaves fixed the ∞^2 points of the plane is given by (3). This is the non-contact factor of our larger group (1). An important subgroup of this is

$$X = x,\ Y = y,\ \Theta = a\theta + b, \tag{4}$$

where $a(\neq 0)$ and b are arbitrary constants. This may be defined as that subgroup of (1) which leaves the ∞^2 points fixed and has the *same* effect on each point of the plane.

This group (4) contains two important types of lineal element transformations: (1) the *turn* which has been considerably studied in our preceding work,[1] and (2) the *magnicline*, a new type, which we are introducing in the present paper. A turn T_α is the result of rotating each lineal element about

its point through a constant angle α, whereas a magnicline M_m is the effect produced by multiplying the inclination θ of each lineal element by the same constant number $m \neq 0$ while leaving its point fixed.

It is observed that a turn may be considered to be an additive correspondence, while a magnicline is a multiplicative correspondence. Of course, both of these operations are contained in the larger group (4). Any correspondence of this group may be factored into a magnicline by a turn (or a turn by a magnicline) in one and only one way.

4. Isogonal and Multiplicative Trajectories.—Applied to a field of lineal elements F, the operation T_α produces ∞^1 isogonal trajectories, and the operation M_m produces ∞^1 multiplicative trajectories. By varying α, *the complete family of* ∞^2 *isogonal trajectories* of $F: \theta = \theta(x, y)$ is

$$y'' = (1 + y'^2)(\theta_x + y'\theta_y), \tag{5}$$

whereas by varying m, the *complete family of* ∞^2 *multiplicative trajectories is*

$$y'' = (1 + y'^2)(\arctan y')\left(\frac{\theta_x}{\theta} + y'\frac{\theta_y}{\theta}\right). \tag{6}$$

For example, consider the family of ∞^1 circles concentric at the origin: $x^2 + y^2 = \text{const.}$ The isogonal trajectories are the equi-angular spirals

$$\cos\alpha \log(x^2 + y^2) + 2\sin\alpha \arctan\frac{y}{x} = \text{const.}, \tag{7}$$

and the multiplicative trajectories are

$$(x^2 + y^2)^{n/2} \cos n\left[\arctan\frac{y}{x} - \frac{\pi}{2}\right] = \text{const.} \tag{8}$$

(If $\alpha = \pi/2$, then (7) represents the ∞^1 straight lines through the origin; whereas when $n = -1$ (8) represents the ∞^1 circles tangent to the x-axis at the origin). Note that $m + n = 1$.

Finally applying the total transformation (4) to a field F, we obtain the ∞^1 *additive-multiplicative trajectories* of F. By varying a and b, we obtain the *complete system of* ∞^3 *additive-multiplicative trajectories* which is given by the differential equation of third order

$$y''' = \frac{(\theta_{xx} + 2y'\theta_{xy} + y'^2\theta_{yy})}{\theta_x + y'\theta_y}y'' + \left(\frac{\theta_y}{\theta_x + y'\theta_y} + \frac{2y'}{1 + y'^2}\right)y''^2. \tag{9}$$

This differential equation is a special case of the more general differential equation

$$y''' = G(x, y, y')y'' + H(x, y, y')y''^2, \tag{10}$$

which I have studied very extensively.[2] Thus we may state the

THEOREM. *In order that a system S of* ∞^3 *curves represents the additive-multiplicative trajectories of a given field, it is necessary (but not sufficient) that S possesses the dynamical Property I; this may be stated simply in either of these alternative geometric forms:*

Property I_a: *For the* ∞^1 *curves of S passing through any lineal element E of the plane, construct the osculating parabolas at E. The locus of the foci of these is a circle passing through the point of E.*

Property I_b: *The* ∞^1 *directrices of these osculating parabolas all pass through a fixed point, thus forming a pencil of straight lines.*

Differential equations of the general form (10), that is, triply infinite systems of curves possessing Property I, have presented themselves in a great variety of physical and geometrical problems. For example, dynamical trajectories, catenaries, brachistochrones, sectional families and curvature trajectories.[2]

[1] Kasner, "The Group of Turns and the Slides and the Geometry of Turbines," *Am. Jour. Math.*, **33**, 193–202 (1911).

Kasner and De Cicco, "Geometry of Turbines, Flat Fields and Differential Equations," *Ibid.*, **59**, 545–563 (1937).

De Cicco, "The Geometry of Fields of Lineal Elements," *Trans. Amer. Math. Soc.*, **47**, 207–229 (1940).

[2] Kasner, "The Trajectories of Dynamics," *Ibid.*, **7**, 401–424 (1906). See my Princeton Colloquium Lectures, "Differential-Geometric Aspects of Dynamics" (1912), American Mathematical Society, reprinted 1934, and many later papers.

LINEAL ELEMENT TRANSFORMATIONS WHICH PRESERVE THE DUAL-ISOTHERMAL CHARACTER

BY JOHN DE CICCO

DEPARTMENT OF MATHEMATICS, ILLINOIS INSTITUTE OF TECHNOLOGY, CHICAGO, ILL.

Communicated July 8, 1941

1. *Introduction.*—In the preceding paper, Kasner has proved the following result. The group of lineal element transformations which convert all isothermal families into isothermal families is given in cartesian coördinates (x, y, θ), where θ is the inclination, by $X = \phi(x, y)$, $Y = \psi(x, y)$, $\Theta = a\theta + h(x, y)$ where ϕ and ψ satisfy the Cauchy-Riemann equations (direct or reverse), h is any harmonic function of (x, y) and a is a non-zero constant.

We propose to obtain an analogue of this result in equilong geometry. Under the equilong group, any dual-isothermal family is carried into any dual-isothermal family. In this paper, we shall determine the complete group of lineal element transformations which send every dual-isothermal family into a dual-isothermal family.

A lineal element consists of a point and a direction through that point. In this work, it is found most convenient to define any lineal element E of the plane by the hessian coördinates (u, v, w), where v is the perpendicular distance from the origin to the line of E, u is the angle from the x-axis to this perpendicular and w is the distance from the foot of the perpendicular to the point of E.[1] Kasner has called the totality of ∞^3 lineal elements of the plane the *opulence*.[2]

Kasner has termed a field *dual-isothermal* if its ∞^1 integral curves are equilongly equivalent to the points of a straight line (a pencil of points). The field $w = w(u, v)$ is dual-isothermal if and only if w is dual-harmonic, that is, w satisfies the analogue of the Laplace equation $w_{vv} = 0$. Thus w is of the form $w = v\alpha(u) + \beta(u)$. The integral curves of a dual-isothermal field are $v\lambda(u) + \mu(u) = \text{const.}$ Therefore a dual-isothermal family is given by $F(u, v) = \text{const.}$ where F is a dual-harmonic function.

Obviously any lineal element transformation carries any field into a field. But it does not preserve the dual-isothermal character. Our fundamental result is that *the complete group of lineal element transformations which convert all dual-isothermal families into dual-isothermal families is* $U = \phi$, $V = (b_1v + b_2w + b_3)/(a_1v + a_2w + a_3)$, $W = (c_1v + c_2w + c_3)/(a_1v + a_2w + a_3)$, *where* $\phi, a_1, a_2\ a_3, b_1, b_2, b_3, c_1, c_2, c_3$ *are functions of* u *only*. There are $\infty^{9f(1)}$ transformations in our group.[3]

The contact part of this group is $U = \phi(u)$, $V = v\psi(u) + \chi(u)$. Thus the contact transformations which preserve dual-isothermal character constitute a group, much larger than the equilong group.

2. Dual-Isothermal Families into Dual-Isothermal Families.—Under the arbitrary lineal element transformation

$$U = \phi(u, v, w),\ V = \psi(u, v, w),\ W = \chi(u, v, w), \tag{1}$$

let the field $w = w(u, v)$ be carried into the field $W = W(U, V)$.

By taking the partial derivatives of (1), we can easily show that W_U and W_V are given by

$$W_U = \frac{d + ew_u + fw_v}{a + bw_u + cw_v},\quad W_V = \frac{h + kw_u + lw_v}{a + bw_u + cw}, \tag{2}$$

where

$$a = \phi_u\psi_v - \phi_v\psi_u,\ b = \phi_w\psi_v - \phi_v\psi_w,\ c = \phi_u\psi_w - \phi_w\psi_u;$$

$$d = \chi_u\psi_v - \chi_v\psi_u,\ e = \chi_w\psi_v - \chi_v\psi_w,\ f = \chi_u\psi_w - \chi_w\psi_u; \tag{3}$$

$$h = \phi_u\chi_v - \phi_v\chi_u,\ k = \phi_w\chi_v - \phi_v\chi_w,\ l = \phi_u\chi_w - \phi_w\chi_u.$$

Upon taking the partial derivative of the second of equations (2) with respect to V, we find

$$(a + bw_u + cw_v)^2 W_{VV} =$$
$$(\phi_u + \phi_w w_u)[(a + bw_u + cw_v)(h_v + k_v w_u + l_v w_v) - (h + kw_u + lw_v)$$
$$(a_v + b_v w_u + c_v w_v)] - (\phi_v + \phi_w w_v)[(a + bw_u + cw_v)(h_u + k_u w_u +$$
$$l_u w_v) - (h + kw_u + lw_v)(a_u + b_u w_u + c_u w_v)] + (\phi_u + \phi_w w_u)[\{(ak -$$
$$bh) - (bl - ck)w_v\}w_{uv} + \{(al - ch) + (bl - ck)w_u\}w_{vv}] - (\phi_v +$$
$$\phi_w w_v)[\{(ak - bh) - (bl - ck)w_v\}w_{uu} + \{(al - ch) +$$
$$(bl - ck)w_u\}w_{uv}]. \quad (4)$$

Suppose now every dual-isothermal field is carried into a dual-isothermal field. Then upon setting $w_{vv} = 0$ and $W_{VV} = 0$, the preceding equation must be an identity in w_u, w_v, w_{uu} and w_{uv}.

Upon setting the coefficients of w_{uu} and w_{uv} equal to zero, we obtain the following equations

$$\phi_w(bl - ck) = 0, \phi_v(ak - bh) = 0,$$
$$\phi_v(bl - ck) - \phi_w(ak - bh) = 0,$$
$$\phi_u(bl - ck) + \phi_w(al - ch) = 0,$$
$$\phi_u(ak - bh) - \phi_v(al - ch) = 0. \quad (5)$$

Let us suppose that $\phi_w \neq 0$. The preceding equations are equivalent to

$$bl - ck = 0, ak - bh = 0, al - ch = 0. \quad (6)$$

Suppose now $b = c = 0$. From (3), it follows since $\phi_w \neq 0$ that $\psi_u : \phi_u = \psi_v . \partial\phi_v = \psi_w : \phi_w$. This makes the jacobian J of our lineal element transformation vanish. Hence we conclude that either b or $c \neq 0$.

Since either b or c is not zero, from (6) we find $h:a = k:b = l:c = \rho$. The value of the jacobian J is

$$J = \begin{vmatrix} \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{vmatrix} = -h\psi_w + k\psi_u + l\psi_v = \rho(-a\psi_w + b\psi_u + c\psi_v). \quad (7)$$

Substituting from (3) into the last parenthesis, we find that $J = 0$. This is impossible. We therefore conclude that $\phi_w = 0$.

Since $\phi_w = 0$, the equations (5) become

$$\phi_u(bl - ck) = 0, \phi_v(bl - ck) = 0, \phi_v(ak - bh) = 0, \phi_u(ak - bh) -$$
$$\phi^v(al - ch) = 0. \quad (8)$$

Suppose now that $\phi_v \neq 0$. The preceding equations then reduce to (6). As we have already seen that the system (6) is impossible, we conclude that $\phi_v = 0$. Therefore *both ϕ_v and ϕ_w are zero.*

Since $\phi_v = \phi_w = 0$ and $\phi_u \neq 0$, our equations (5) and (10) are identically satisfied. Therefore *the equations (5) will be identically satisfied if and only if our lineal element transformation is of the form*

$$U = \phi(u),\ V = \psi(u, v, w),\ W = \chi(u, v, w). \tag{9}$$

The transformation (9) preserves the set of linear flat fields $u = \text{const.}$ Hence if it is also to carry every dual-isothermal field into a dual-isothermal field, our transformation (9) must carry every linear turbine into a linear turbine since the intersection of a linear flat field and a dual-isothermal field is a linear turbine (see references 1 and 2). Thus the correspondence (9) must preserve the totality of ∞^2 linear turbines $w = \alpha v + \beta$ of any arbitrary linear flat field $u = \text{const.}$ We may therefore conclude by the well-known theorem on the projective group of the plane that the functions ψ and χ must be linear fractional functions of v and w with coefficients functions of u only.

FUNDAMENTAL THEOREM. *The complete group of lineal element transformations which convert all dual-isothermal families into dual-isothermal families is*

$$U = \phi,\quad V = \frac{b_1 v + b_2 w + b_3}{a_1 v + a_2 w + a_3},\quad W = \frac{c_1 v + c_2 w + c_3}{a_1 v + a_2 w + a_3}, \tag{10}$$

where ϕ, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 are functions of u only. This is also the entire group of lineal element transformations which preserve the totality of ∞^3 linear turbines.

This group contains the *whirl-motion group* G_6, studied quite extensively by Kasner and the author (see references 1 and 2).

It can be proved that the contact part of this group (10) is

$$U = \phi(u),\ V = v\psi(u) + \chi(u). \tag{11}$$

This is much larger than the equilong group. Of course, the equilong group is the above where $\psi = \pm\phi_u$.

It is of interest that this result obtained in the equilong theory differs very essentially from the corresponding result obtained by Kasner in the conformal theory.

[1] Kasner, "The Group of Turns and Slides and the Geometry of Turbines," *Am. Jour. Math.*, **33**, 193–202 (1911).

Kasner and De Cicco, "Geometry of Turbines, Flat Fields and Differential Equations," *Ibid.*, **59**, 545–563 (1937).

[2] De Cicco, "The Geometry of Fields of Lineal Elements," *Trans. Amer. Math. Soc.*, **47**, 207–229 (1940).

[3] Kasner in a review of Riquier's treatise has suggested a notation for the content of manifolds depending on arbitrary functions. See *Bull. Amer. Math. Soc.*, 1912. According to this notation, the content of our group is $\infty^{9f(1)}$ whereas the content of Kasner's group is $\infty^{1+4f(1)}$.

NON-LINEAR EQUATIONS OF THE ELECTROMAGNETIC FIELD

BY ROBERT E. STREET

RENSSELAER POLYTECHNIC INSTITUTE

Communicated June 16, 1941

The recent theory of the electromagnetic field of Born and Infeld[1] and the earlier theory of Whittaker[2] can be easily combined to give a formal derivation of a very general set of field equations, which contain not only these two theories as special cases but also the older theories of Maxwell and Mie.

Consider any region of space-time, devoid of matter, but containing a field specified by the skew symmetric tensor f_{kl}, $k, l = 1, 2, 3, 4$. Let the metric for this space be the usual one of relativity

$$ds^2 = g_{\lambda\mu} dx^\lambda dx^\mu, \ \lambda, \mu = 1, 2, 3, 4,$$

and let g = the determinant $|g_{kl}|$. We replace the usual relation between the tensor f_{kl} and the vector potential function ϕ_k by

$$f_{kl} = \frac{\partial \phi_k}{\partial x^l} - \frac{\partial \phi_l}{\partial x^k} + q_{kl}, \tag{1}$$

where the q_{kl} are the six components of a skew symmetric tensor and functions of the f_{kl}. We assume the tensor q_{kl} can be expanded in powers of the f_{kl} beginning with terms of the second degree.

Using the comma notation for covariant differentiation, which reduces to ordinary differentiation when the metric is the one of special relativity for rectangular coördinates, we take the divergence of (1) obtaining the first set of field equations

$$f^{*k\alpha}, \alpha = q^{*k\alpha}, \alpha. \tag{2}$$

Here we use the tensors dual to f_{kl} and q_{kl} defined by

$$f^{*kl} = \frac{1}{2\sqrt{-g}} \epsilon^{kl\alpha\beta} f_{\alpha\beta},$$

where ϵ^{klmn} is zero unless $klmn$ are all different and then ± 1 according as $klmn$ form an even or odd permutation of 1234.[3]

This set of field equations gives the divergence of the field tensor equal to an "apparent" magnetic current $n^k = q^{*k\alpha}, \alpha$, which is a function of the field when the q^{*kl} are expanded in powers of the f_{kl} and satisfies the conservation law

$$\frac{\partial}{\partial x^\alpha} (\sqrt{-g}\, n^\alpha) = 0.$$

To obtain the second set of field equations and the equations of motion, we use the assumption that a scalar function l of the tensor f_{kl} exists such that the integral

$$J = \int_D L dx,\ L = \sqrt{-g}l,\ dx = dx^1 dx^2 dx^3 dx^4, \tag{3}$$

taken over any regular domain D of the four space is invariant under an arbitrary change in the values of the functions ϕ_k or in the coördinates allowed by the metric.[4] An acceptable function l can be written as a function of F and G^2, the only invariants which can be formed from the single tensor f_{kl} where

$$F = \frac{1}{4} f_{\lambda\mu} f^{\lambda\mu},\ G = \frac{1}{4} f^*_{\lambda\mu} f^{\lambda\mu}.$$

Allowing variations in the potential functions ϕ_k but with no change in the coördinates the invariance of (3) is given by

$$\delta J = \int_D \delta L dx = 0.$$

Straightforward calculation gives

$$\delta L = \sqrt{-g}\left[p^{\alpha\beta} \frac{\partial \delta\phi_\alpha}{\partial x^\beta} + \frac{1}{2} p^{\alpha\beta} \frac{\partial q_{\alpha\beta}}{\partial f_{\lambda\mu}} \frac{\partial \delta\phi_\lambda}{\partial x^\mu} + \frac{1}{4} p^{\alpha\beta} \frac{\partial q_{\alpha\beta}}{\partial f_{\lambda\mu}} \frac{\partial q_{\lambda\mu}}{\partial f_{\rho\sigma}} \frac{\partial \delta\phi_\rho}{\partial x^\sigma} + \ldots\ldots \right]$$

with $p^{kl} = \dfrac{\partial l}{\partial f_{kl}}$. Substituting in the integral for δJ and integrating by parts we have

$$\int_D \left[\frac{\partial}{\partial x^\beta}\left(\sqrt{-g}\, p^{\alpha\beta}\right) + \frac{1}{2}\frac{\partial}{\partial x^\beta}\left(\sqrt{-g}\, p^{\lambda\mu} \frac{\partial q_{\lambda\mu}}{\partial f_{\alpha\beta}}\right) + \frac{1}{4}\frac{\partial}{\partial x^\beta}\left(\sqrt{-g}\, p^{\lambda\mu} \frac{\partial q_{\lambda\mu}}{\partial f_{\rho\sigma}} \frac{\partial q_{\rho\sigma}}{\partial f_{\alpha\beta}}\right) + \ldots. \right] \delta\phi_\alpha dx = 0,$$

if we take the $\delta\phi_k$ arbitrary within D but zero on the boundary S of D. Since the $\delta\phi_k$ are arbitrary, the continuous integrand must vanish everywhere giving us our second set of field equations

$$p^{k\alpha},\ \alpha + m^{k\alpha},\ \alpha = 0,\ k = 1, 2, 3, 4 \tag{4}$$

$$m^{kl} \equiv \frac{1}{2} p^{\alpha\beta} \frac{\partial q_{\alpha\beta}}{\partial f_{kl}} + \frac{1}{4} p^{\alpha\beta} \frac{\partial q_{\alpha\beta}}{\partial f_{\lambda\mu}} \cdot \frac{\partial q_{\lambda\mu}}{\partial f_{kl}} + \ldots..$$

If the presence of matter in the field is indicated by a current vector with components j^k the field equations (4) become

$$p^{k\alpha},\ \alpha + m^{k\alpha},\ \alpha = j^k,\ k = 1, 2, 3, 4, \tag{5}$$

while (2) will not change.

Equations (5) are equivalent to the ones derived by Born[5] where he used the tensor m^{kl} to represent the electric and magnetic moments of the particles. Here we have m^{kl} even in a pure field with no matter, indicating an "apparent" moment due to the non-linear character of the field.

We next allow an infinitesimal transformation in the coördinates

$$\bar{x}^k = x^k + \epsilon\, \eta^k(x)$$

with parameter ϵ whose second and higher degree powers are neglected in what follows while the $\eta^k(x)$ are arbitrary differentiable functions of the x^k. The variations of any function $\psi(x)$ are defined by

$$\delta'\psi = \bar{\psi}(\bar{x}) - \psi(x),\ \delta\psi = \bar{\psi}(x) - \psi(x).$$

Then the vanishing of the first variation $\delta' J$ is given by[6]

$$\int_D \delta L dx + \epsilon \int_n \frac{\partial}{\partial x^\alpha}\left(L\eta^\alpha\right)dx = 0. \tag{6}$$

If the symmetric stress tensor T^{kl} is defined by

$$T^{kl} = \frac{1}{\sqrt{-g}}\frac{\partial L}{\partial g_{kl}},\ p^{kl} = \frac{1}{\sqrt{-g}}\frac{\partial L}{\partial f_{kl}},$$

we have

$$\delta L = \frac{1}{2}\sqrt{-g}\left(p^{\alpha\beta}\,\delta f_{\alpha\beta} + T^{\alpha\beta}\,\delta g_{\alpha\beta}\right).$$

Calculating $\delta f_{\alpha\beta}$ and $\delta g_{\alpha\beta}$ we reduce (6) to

$$\int\left[\left(p^{\alpha\beta} f_{\lambda\beta} + T^{\alpha\beta} g_{\lambda\beta}\right)\frac{\partial\eta^\lambda}{\partial x^\alpha} + \frac{1}{2}\left(p^{\alpha\beta}\frac{\partial f_{\alpha\beta}}{\partial x^\lambda} + T^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\right)\eta^\lambda\right]\sqrt{-g}\,dx = \int_D \frac{\partial}{\partial x^\alpha}\left(L\eta^\lambda\delta^\alpha_\lambda\right)dx,$$

or

$$\int_D \frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\ V^\alpha_\lambda\eta^\lambda\right)dx + \int_D t_\lambda\eta^\lambda\sqrt{-g}\,dx = 0 \tag{7}$$

where

$$V^k_l \equiv T^k_l + p^{k\alpha}f_{l\alpha} - L\delta^k_l$$

$$\sqrt{-g}\,t_k \equiv p^*_{k\alpha}\frac{\partial\sqrt{-g}\,f^{*\alpha\beta}}{\partial x^\beta} + f_{k\alpha}\frac{\partial\sqrt{-g}\,p^{\alpha\beta}}{\partial x^\beta} - \frac{\partial\sqrt{-g}\,T^\alpha_k}{\partial x^\alpha}.$$

Here we used the identity

$$\frac{1}{2}\sqrt{-g}\,p^{\alpha\beta}\left(\frac{\partial f_{\alpha\beta}}{\partial x^k}+\frac{\partial f_{\beta k}}{\partial x^\alpha}+\frac{\partial f_{k\alpha}}{\partial x^\beta}\right)\equiv p^*_{k\alpha}\frac{\partial\sqrt{-g}\,f^{*\alpha\beta}}{\partial x^\beta}.$$

Suppose the η^k are arbitrary inside D but zero on the boundary S. Then the first integral in (7) must vanish since it can be written as a surface integral over S by the divergence theorem. It follows that t_k must vanish everywhere or we have the "conservation" law[7]

$$\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\,T^\alpha_k\right)=f_{k\alpha}\frac{\partial}{\partial x^\beta}\left(\sqrt{-g}\,p^{\alpha\beta}\right)+p^*_{k\alpha}\frac{\partial}{\partial x^\beta}\left(\sqrt{-g}\,f^{*\alpha\beta}\right). \quad (8)$$

To find T^l_k we return to (7). Since t_k is zero we must also have the integrand in the first integral zero or

$$\sqrt{-g}\,V^\alpha_\lambda\frac{\partial\eta^\lambda}{\partial x^\alpha}+\eta^\lambda\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\,V^\alpha_\lambda\right)=0.$$

Since η^λ and $\dfrac{\partial\eta^\lambda}{\partial x^\alpha}$ are independent of each other

$$V^k_l=0,\ \frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\,V^\alpha_k\right)=0$$

giving (8) again and also

$$T^k_l=L\delta^k_l-p^{k\alpha}f_{l\alpha}. \quad (9)$$

The symmetry of the tensors T^{kl} and T_{kl} together with (9) gives the condition

$$p^\alpha_k f_{l\alpha}=f^\alpha_k p_{l\alpha}.$$

We can derive (8) from (9) by use of the field equations if we arbitrarily choose (9) as the definition of T^k_l, without the use of the variation produced in J by an infinitesimal transformation. But the use of the variational principle leads naturally to (9) as well as showing the symmetry of T^{kl} and guaranteeing the invariance of J under changes in the coördinate system.

Now (8) has been derived on the assumption of a pure field in empty space with no matter present. In the theories of Maxwell and Born such a field has the equations

$$p^{k\alpha},\ \alpha=0 \qquad f^{*k\alpha},\ \alpha=0$$

and (8) reduces to the true conservation law $T^\alpha_k,\ \alpha=0$. Our field equations give

$$T^\alpha_k,\ \alpha=p^*_{k\alpha}n^\alpha-f_{k\alpha}m^{\alpha\beta},\ \beta$$

which does not vanish even in a pure field.

When actual matter of mass density μ_0 and charge density ρ_0 is present, equation (5) indicates that we take as our force equation

$$\mu_0 \frac{dv_k}{ds} = f_{k\alpha} j^\alpha + p^*_{k\alpha} n^\alpha + f_{k\alpha} j'^\alpha.$$

The first term on the right can be interpreted as the force on the matter of charge ρ_0; the second term is the force on an apparent magnetic charge of density σ_0 if σ_0 is defined by $n^k = \sigma_0 w^k$ where w^k is a "velocity" vector; the third term is the force on an apparent electric charge ρ_0' due to the moment tensor m^{kl}. The current vector j'^k is $j'^k = m^{k\alpha}$, α according to the usual relationship between electric current and the moment tensor. Such quantities with the dimensions of force were to be expected in a non-linear field.

We have given here a quite formal but nevertheless concise derivation of a general set of non-linear field equations for the electromagnetic field. If in (1) the components q_{kl} are all identically zero, the original theory of Born results while the introduction of the tensor q_{kl} leads to the equations (5), which he obtained in a different manner without introducing this tensor. By retaining q_{kl} but taking as l the Lagrangian of Maxwell's theory we have the theory of Whittaker.

A partial solution for the simplest case of a centrally symmetric electrostatic field has been worked out. It is similar to Born's solution of the same case but the lack of a true potential function does not allow for a complete treatment.

[1] Born, M., and Infeld, L., *Proc. Roy. Soc.*, (A) **144,** 425–451 (1934) and many subsequent papers. See the bibliography in Born, M., *Annales de l'Inst. H. Poincare*, **7,** 155 (1938).

[2] Whittaker, E. T., *Proc. Roy. Soc.*, (A) **113,** 496–511 (1926–1927).

[3] Veblen, O., *Invariants of Quadratic Differential Forms*, Cambridge, 1927.

[4] In such a way does the calculus of variations enter physics as a Lagrangian principle! The Euler equations, which are necessary for the vanishing of the first variation, are taken for the equations of physics but nothing is said about the second variation or sufficient conditions.

[5] Born, M., *Memoirs Ind. Inst. of Sci.*, **3,** No. 1, 9–10 (1936).

[6] Pauli, W., *Relativitatstheorie*, Teubner, 1921, pp. 616–618.
Weyl, H., *Raum—Zeit—Materie*, 5th Ed., Springer, 1923, pp. 233–235.

[7] For a discussion of conservation laws and references to the literature see Weyl, loc. cit., Chap. IV.